Standard XI

Mathematics & Statistics

Commerce Part 1

The Coordination Committee formed by GR No. Abhyas - 2116/(Pra.Kra.43/16) SD - 4 Dated 25.4.2016 has given approval to prescribe this textbook in its meeting held on 20.06.2019 and it has been decided to implement it from the educational year 2019-20.

MATHEMATICS AND STATISTICS

(COMMERCE)

Part-I

STANDARD XI

2019

Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune - 411 004

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The Constitution of India

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

NATIONAL ANTHEM

Jana-gana-mana-adhināyaka jaya hē
Bhārata-bhāgya-vidhātā,

Panjāba-Sindhu-Gujarāta-Marāthā
Drāvida-Utkala-Banga

Vindhya-Himāchala-Yamunā-Gangā
uchchala-jaladhi-taranga

Tava subha nāmē jāgē, tava subha āsisa māgē,
gāhē tava jaya-gāthā,

Jana-gana-mangala-dāyaka jaya hē
Bhārata-bhāgya-vidhātā,

Jaya hē, Jaya hē, Jaya hē,
Jaya jaya jaya, jaya hē.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect, and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness.

PREFACE

Dear Students,

Welcome to the eleventh standard!

You have successfully completed your secondary education and have entered the higher secondary level. You will now need to learn certain mathematical concepts and acquire some statistical skills to add accuracy and precision to your work. Maharashtra State Bureau of Text Book Production and Curriculum Research has modified and restructured the curriculum in Mathematics and Statistics for the Commerce stream in accordance with changing needs of the society.

The curriculum of Mathematics and Statistics is divided in two parts. Part-1 covers topics in Algebra, Co-ordinate Geometry, Complex Numbers, Sets and Relations. Functions and Calculus. Part-2 covers Combinatorics and Statistics. There is a special emphasis on applications. Activities are added at the end of chapters for creative thinking. Some material will be made available on E-balbharati website (ebalbharati.in). It contains a list of specimen practical problems on each chapter. Students should complete the practical exercises under the guidance of their teachers. Journals are to be maintained by students and assessed by teachers.

You are encouraged to use modern technology in your studies. Explore the Internet for more recent information on topics in the curriculum. Get more examples and practice-problems from the Internet. You will enjoy learning if you follow three simple principles: a thorough study of the textbook, learning based on activities, and continuous practice of solving problems.

On the title page Q.R. code is given. It will help you to get more knowledge and clarity about the contents.

This textbook is prepared by mathematics subject commitee and study group. This book has also been reviewed by senior teachers and eminent scholars. The Bureau would like to thank all of them for their contribution in the form of creative writing, constructive criticism, and valuable suggestions in making this book useful. Also the Bureau is grateful to members of the mathematics subject commitee, study group and reviewers for sparing their valuable time in preparing this book. The Bureau hopes that the textbook will be received well by all users in the right spirit.

You are now ready to study. Best wishes for a happy learning experience.

(Dr. Sunil Magar)
Director
Maharashtra State Bureau of Textbook
Production and Curriculum Research, Pune.

Pune
Date : 20 June 2019
Indian Solar Date : 30 Jyeshtha 1941

XI Mathematics Commerce Part I

Competency statements

Sr. No	Area	Topic	Competency Statements
1	Sets and Relations	Sets	The student will be able to • work with sets and set functions. • construct sets from given conditions/description/ rule. • solve problems using set theory.
		Relations	• identify the types of relations. • use relations to associate different sets. • verify equality, equivalence or other relation-ships between given sets.
2	Functions	Functions	• work with function defined on different domains. • identify different types of functions. • carry out complicated operations on functions.
3	Complex Numbers	Complex Numbers	• simplify algebraic expressions involving complex numbers.
4	Sequence and series	Sequence	• identify the type of a given sequence. • find the general term of given sequence.
		Series	• identify the type of a given series • find the n^{th} term of a given series • find the sum of the first n terms of a given series • find the sum to infinite terms of a given series
5	Locus and Straight Line	Locus and Straight Line	• find equation of a straight line satisfying given conditions • identify properties of given set of straight lines
6	Determinats	Determinants	• find value of a determinant. • simplify determinant. • solve linear equations in 2/3 variables, find area of triangle using determinants.
7	Limits	Limits	• find limit of a function • determine whether a given function has a limit
8	Continuity	Continuity	• determine whether a given function is continuous at a given point • determine whether a given function is continuous over a specified interval • identify points of discontinuity of a given function
9	Differentiation	Differentiation	• differentiate algebraic functions

INDEX

1. SETS AND RELATIONS

Let's study.

- Representation of a Set
- Intervals
- Types of sets
- Operations on sets
- Ordered pair
- Relations

Let's recall.

The concept of a set was developed by German mathematician George Cantor (1845–1918)

You have already learnt about sets and some basic operations involving them in the earlier standards.

In everyday life, we generally talk about group or a collection of objects. Surely you must have used the words such as team, bouquet, bunch, flock, family for collection of different objects.

It is very important to determine whether a given object belongs to a given collection or not. Consider the following collections:

i) Successful persons in your city.

ii) Happy people in your town.

iii) Clever students in your class.

iv) Days in a week.

v) First five natural numbers.

First three collections are not examples of sets, but last two collections represent sets. This is because in first three collections, we are not sure of the objects. The terms 'successful persons,' 'Happy people', 'Clever student' are all relative terms. Here, the objects are not well–defined. In the last two collections. We can determine the objects clearly. Thus, we can say that objects are well–defined.

1.1 SET:

Definition:

A Collection of well–defined objects is called a set.

Object in a set is called its element or member.

We denote sets by capital letters A,B,C. etc. The elements of a set are represented by small letters a, b, c, x, y, z etc. If x is an element of a set A we write $x \in$ A, and read as 'x belongs to A'. If x is not an element of a set A, we write $x \notin$ A, and read as 'x does not belong to A.'

For example, zero is a whole number but not a natural number.

$\therefore$ $0 \in W$ and $0 \notin N$

Representation of a set:

1) Roster method:

In the Roster method, we list all the elements of the set within brackets, and separate the elements by commas.

Example : State the sets using Roster method.

i) B is the set of all days in a week.

B = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

ii) C is the set of all vowels in English alphabets.

$C = \{a, e, i, o, u\}$

2) Set-Builder method:

In the set builder method, we describe the elements of the set by specifying the property which determines the elements of the set uniquely.

Example : State the sets using set–builder method.

i) Y is the set of all months of a year.

Y = {x | x is month of a year}

ii) B is the set of perfect squares of natural numbers.

B = {x∈N/x is perfect square}

3) Venn Diagram:

The pictorial representation of a set is called Venn diagram. Generally, the geometrical closed figures like circle, triangle or rectangle, are used to represent the sets, which are known as Venn diagrams and are named after the English logician John Venn.

In Venn diagram the elements of the sets are shown as points enclosed in the diagram representing set:

A = {1,2,3} B = {a,b,c,d,e,f} C = {4,5,6}

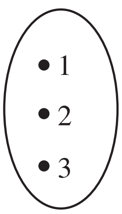


Fig. 1.1

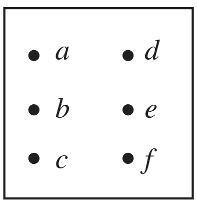


Fig. 1.2

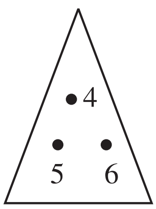


Fig. 1.3

Let's Note.

1) If the elements are repeated, write them once.

2) While listing the elements of a set, the order in which the elements are listed is immaterial.

Let's learn.

1.2 INTERVALS:

1) **Open Interval:** Let a, b ∈R and a < b then the set {x / x ∈ R a < x < b} is called open interval and is denoted by (a,b). All the numbers between a and b belong to the open interval (a,b) but a, b themselves do not belong to this interval.

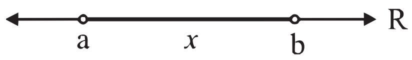


Fig. 1.4

∴ (a,b) = {x / x ∈ R, a < x < b}

2) **Closed Interval:** Let a, b∈R and a<b then the set {x / x ∈ R a ≤ x ≤ b} is called closed interval and is denoted by [a,b]. All the numbers between a and b belong to the closed interval [a, b]. Also a and b belong to this interval.

Fig. 1.5

[a, b] = {x / x ∈ R, $a \le x \le b$}

3) **Semi-closed Interval:**

[a, b) = {x/x∈R, $a \le x < b$}

Fig. 1.6

Note that a∈[a, b] and b ∉ [a, b]

4) **Semi–open Interval:**

(a, b] = {x / x ∈ R, $a < x \le b$}

Fig. 1.7

(a, b] excludes a but includes b.

5) i) The set of all real numbers greater than a i.e. $(a,\infty) = \{x/x \in R,\ x > a\}$

Fig. 1.8

ii) The set of all real numbers greater than or equal to a

i.e. $[a,\infty) = \{x/x \in R,\ x \geq a\}$

Fig. 1.9

6) i) The set of all real numbers less than b. ie. $(-\infty, b)$

Fig. 1.10

$\therefore (-\infty,\ b) = \{x/x \in R,\ x < b\}$

ii) The set of all real numbers less than or equal to b i.e. $(-\infty, b]$

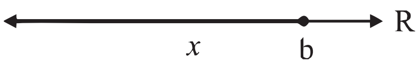


Fig. 1.11

$\therefore (-\infty, b] = \{x/x \in R,\ x \leq b\}$

7) The set of all real numbers i.e. $(-\infty, \infty)$

Fig. 1.12

$\therefore (-\infty,\infty) = \{x/x \in R,\ -\infty < x < \infty\}$

Number of elements of a set: (Cardinality)

The number of distinct elements contained in a finite set A is denoted by n (A).

Thus, if A = {1,2,3,4}, then n (A) = 4

1.3 TYPES OF SETS:

1) Empty Set:

A set containing no element is called an empty or a null set and is denoted by the symbol ϕ or { } or void set.

e.g. $A = \{x / x \in N,\ 1 < x < 2\}$, $n(A) = 0$

2) Singleton set:

A Set containing only one element is called a singleton set.

e.g. Let A be the set of all integers which are neither positive nor negative.

$\therefore A = \{0\}$, $n(A) = 1$

3) Finite set:

A set in which the process of counting of elements comes to an end is called a finite set.

e.g. the set of letters in the word 'beautiful'.

$A = \{b,e,a,u,t,i,f,l\}$, $n(A) = 8$

A is a infinite set

4) Infinite set:

A set which is not finite, is called an infinite set.

e.g. set of natural numbers.

Let's Note.

1) An empty set is a finite set.
2) N, Z, set of all points on a circle are infinite sets.

Some definitions :

1) Equality of sets:

Two sets are said to be equal if they contain the same elements i.e. if $A \subseteq B$ and $B \subseteq A$.

For example:

Let X be the set of letters in the word 'ABBA' and Y be the set of letters in the word 'BABA'.

$\therefore$ X = {A, B}, Y = {B, A}

Thus the sets X and Y are equal sets and we denote it by X = Y

2) Equivalent sets:

Two finite sets A and B are said to be equivalent if n (A) = n (B)

A = {d, o, m, e}

B = {r, a, c, k}

Here n (A) = n (B)

There for A and B are equivalent sets

3) Subset:

A set A is said to be a subset of set B if every element of A is also an element of B and we write $A \subseteq B$.

4) Superset:

If $A \subseteq B$, then B is called a superset of A and we write, $B \supseteq A$.

5) Proper Subset:

A nonempty set A is said to be a proper subset of the set B, if all elements of set A are in set B and atleast one element of B is not in A.

i.e. If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and we write $A \subset B$.

e.g. 1) Let A = {1,3,5} and B = {1,3,5,7}. Then, every element of A is an element of B but $A \neq B$.

$\therefore A \subset B$, i.e. A is a proper subset of B.

Remark: If there exists even a single element in A which is not in B then A is not a subset of B and we write $A \not\subset B$.

6) Universal set:

If in a particular discussion all sets under consideration are subsets of a set, say U, then U is called the universal set for that discussion.

The set of natural numbers N, the set of integers Z are subsets of set of real numbers R. Thus, for this discussion R is a universal set.

In general universal set is denoted by 'U' or 'X'.

7) Power Set:

The set of all subsets of a given set A is called the power set of A and is denoted by P(A). Thus, every element of power set A is a set.

e.g. consider the set A={a,b}. Let us write all subsets of the set A. We know that ϕ is a subset of every set, so ϕ is a subset of A. Also {a}, {b}, {a,b} are also subsets of A. Thus, the set A has in all four subsets viz. ϕ, {a}, {b}, {a,b}

$\therefore$ P(A) = {ϕ, {a}, {b}, {a,b}}

Operations on sets:

1) Complement of a set:

Let A be a subset of universal set.

The complement of the set A is denoted by

A' or A^c. It is defined as

A' = {x / $x \in$ U $x \notin$ A} = set of all elements in U which are not in A.

Ex 1) Let X = {0, 1, 2, 3, 4, 5, 6, 7, 8} be the Universal Set and

A = {2, 4, 6, 8}

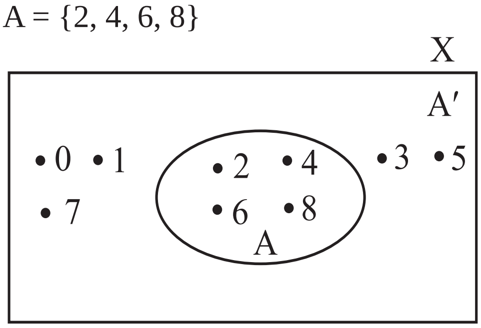


Fig. 1.13

∴ The complement of the set A is

A' = {0, 1, 3, 5, 7}

Properties:

i) (A')' = A

ii) ϕ' = U (U is the universal set)

iii) U' = ϕ

iv) If A $\subseteq$ B then B′ $\subseteq$ A′

2) Union of Sets:

Union of sets A and B is the set of all elements which are in A or in B. (Here 'or' is taken in the inclusive sense)

Thus, A $\cup$ B = {x | x∈A or x∈B}

The union of two sets A and B can be represented by a Venn-diagram Fig. 1.14 and fig. 1.15 represent A $\cup$ B

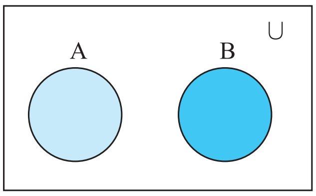


Fig. 1.14
A ∪ B

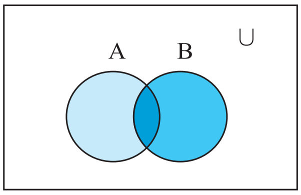


Fig. 1.15
A ∪ B

Properties:

i) A ∪ B = B ∪ A ..... (Commutativity)

ii) (A ∪ B) ∪ C = A ∪ (B ∪ C) .... (Associative Property)

iii) A ∪ ϕ = A ..... (Identity of Union)

iv) A ∪ A = A ...... (Idempotent law)

v) A∪ A' = U

vi) If A⊂B then A∪B = B

vii) U∪A = U

viii) A⊂(A∪B), B⊂(A∪B)

Ex. : let A = {x/x is a prime number less than 10}

B= {x | x is a factor of 8}

find A∪B.

Solution : We have A = {2,3,5,7}

B = {1,2,4,8}

∴ A∪B = {1, 2, 3, 4, 5, 7, 8}

3) Intersection of sets:

The intersection of two sets A and B is the set of all elements which are both in A and B. It is denoted by A∩B.

Thus A∩B = {x | x∈A **and** x∈B}

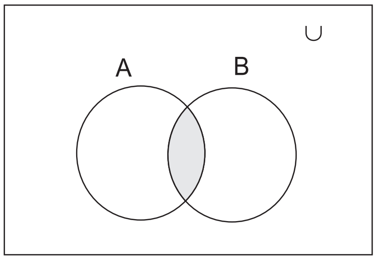


Fig. 1.16
A∩B

The shaded portion in Fig. 1.16 represents the intersection of A and B i.e. A∩B

Properties:

i) A∩B = B ∩A .... Commutativity

ii) (A ∩ B) ∩ C = A ∩ (B ∩ C) .... Associativity

iii) ϕ ∩ A = ϕ

iv) A ∩ A = A .... Idempotent law

v) A ∩ A′ = ϕ

vi) If A⊂B then A∩B =A

vii) U∩A = A

viii) (A∩B) ⊂A, (A∩B) ⊂B

Remark : If A∩B = ϕ, A and B are disjoint sets.

4) Distributive Property

a) $A\cap(B\cup C) = (A\cap B)\cup (A\cap C)$

b) $A\cup(B\cap C) = (A\cup B) \cap (A\cup C)$

DeMorgan's law

If A and B are subsets of a universal set, then

i) $(A\cup B)' = A'\cap B'$

ii) $(A\cap B)' = A'\cup B'$

Ex. 1: If $A=\{1,3,5,7,9\}$ $B = \{1,2,3,4,5,6,7,8\}$ Find $A\cap B$.

Solution: $A\cap B = \{1,3,5,7\}$

Ex. 2: If $A = \{x \,/\, x$ is a factor of $12\}$

$B = \{x \,/\, x$ is a factor of $18\}$

Find $A\cap B$

Solution:

$A=\{1,2,3,4,6,12\}$

$B = \{1,2,3,6,9,18\}$

$\therefore\ A\cap B = \{1,2,3,6\}$

Ex. 3 : If $A = \{1,3,5,7,9\}$

$B = \{2,4,6,8, 10\}$. Find $A\cap B$

Solution: $A\cap B = \{\ \} = \phi$

5) Difference of sets:

Difference of Set A and Set B is the set of elements which are in A but not in B and is denoted by A–B.

The shaded portion in fig. 1.17 represents A–B. Thus, $A-B = \{x|x\in A, x\notin B\}$

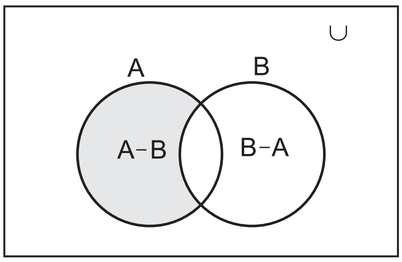


Fig. 1.17

Similary, $B-A = \{y|y\in B, y\notin A\}$

Let's note:

i) A–B is a subset of A and B–A is a subset of B.

ii) The sets A–B, $A\cap B$, B–A are mutually disjoint sets, i.e. the intersection of any of these two sets is the null (empty) set.

iii) $A-B = A\cap B'$

$B-A = A'\cap B$

iv) $A\cup B = (A-B)\cup(A\cap B) \cup (B-A)$

Shaded portion in fig. 1.18 represents $A\cup B$

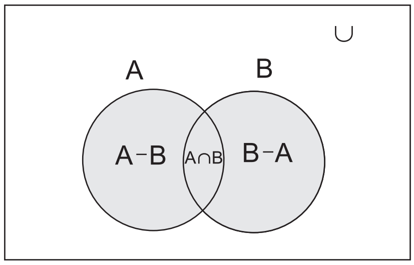


Fig. 1.18 A∪B

Properties of Cardinality of Sets:

For given sets A, B

1) $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

2) When A and B are disjoint sets

$n(A\cup B) = n(A) + n(B)$, as $A\cap B = \phi$,

$\therefore\ n(A\cap B) = 0$

3) $n(A\cap B') + n(A\cap B) = n(A)$

4) $n(A'\cap B) + n(A\cap B) = n(B)$

5) $n(A\cap B') + n(A\cap B) + n(A'\cap B) = n(A\cup B)$

For any sets A, B, C.

6) $n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B)$

$-n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$

7) If $n(A) = m$, $n(P(A)) = 2^m$

where P(A) is the power set of A

8) $n(P') = n(x) - n(P)$

SOLVED EXAMPLES

Ex. 1: If A= {x / x is a factor of 6}

B = {x / x is a factor of 8}

find the A–B and B–A

Solution: A = {1,2,3,6}; B = {1,2,4,8}

$\therefore$ A–B = {3,6}

B–A = {4,8}

Ex.2: $A = \left\{\frac{1}{x}\ x \in N, x < 8\right\}$

$B = \left\{\frac{1}{2x}\ x \in N, x \leq 8\right\}$ Find A–B and B–A

Solution: $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$

$B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right\}$

$\therefore A-B = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$

and $B-A = \left\{\frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right\}$

Ex. 3: If A = {1,2,3,4}, B = {3,4,5,6}

C = {5,6,7,8}, D = {7,8,9,10};

find i) A∪B ii) A∪B∪C iii) B∪C∪D

Are the sets A, B, C, D equivalent?

Solution: We have

i) A∪B = {1,2,3,4,5,6}

ii) A∪B∪C = {1,2,3,4,5,6,7,8}

iii) B∪C∪D = {3,4,5,6,7,8,9,10}

As the number of elements in every set A, B, C, D is 4, the sets A, B, C, D are equivalent.

Ex. 4: Let U = {1,2,3,4,5,6,7,8,9,10}be the universal set, A = {1,3,5,7,9}

B = {2,3,4,6,8,10}, C = {6,7,8,9}

Find i) A' ii) (A∩C)' iii) (A')' iv) (B–C)'

Solution: We have

i) A' = {2,4,6,8,10}

ii) (A∩C) = {7,9}

$\therefore$ (A∩C)' = {1,2,3,4,5,6,8,10}

iii) (A')' = {1,3,5,7,9} = A

iv) B–C = {2,3,4,10}

$\therefore$ (B–C)' = {1,5,6,7,8,9}

Ex. 5: Let X be the universal set, for the non–empty sets A and B, verify the De Morgan's laws

i) (A∪B)' = A'∩B'

ii) (A∩B)' = A'∪B'

Where X = {1,2,3,4,5,6,7, 8,9,10}

A = {1,2,3,4,5} B = {1,2,5,6,7}

Solution: A∪B = {1,2,3,4,5,6,7}

(A∩B) = {1,2,5}

A' = {6,7,8,9,10}

B' = {3,4,8,9,10}

i) (A∪B)' = {8,9,10} ...(1)

A'∩B' = {8,9,10} ...(2)

from (1) and (2), (A∪B)' = A'∩B'

ii) (A∩B)' = {3,4,6,7,8,9,10} ...(3)

A'∪B' = {3,4,6,7,8,9,10} ...(4)

from (3) and (4),

(A∩B)' = A'∪B'

Ex.6: If P = {x/$x^2 + 14x + 40 = 0$}

Q = {x/$x^2 - 5x + 6 = 0$}

R = {x/$x^2 + 17x - 60 = 0$} and the universal set X = {–20, –10 – 4, 2, 3, 4}, find

i) $P \cup Q$ ii) $Q \cap R$ iii) $P \cup (Q \cap R)$

iv) $P \cap (Q \cup R)$

Solution: $P = \{x | x^2 + 14x + 40 = 0\}$

$\therefore P = \{-10, -4\}$

Similarly $Q = \{3, 2\}$, $R = \{-20, 3\}$ and

$X = \{-20, -10, -4, 2, 3, 4\}$

i) $P \cup Q = \{-10, -4, 3, 2\}$

ii) $Q \cap R = \{3\}$

iii) $P \cup (Q \cap R) = \{-10, -4, 3\}$

iv) $P \cap (Q \cup R) = \phi$

Ex. 7: If A and B are the subsets of X and

$n(X) = 50$, $n(A) = 35$, $n(B) = 22$ and

$n(A' \cap B') = 3$, find i) $n(A \cup B)$ ii) $n(A \cap B)$

$n(A' \cap B)$ iv) $n(A \cup B')$

Solution:

i) $n(A \cup B) = n(X) - n(A \cup B)'$

$= n(X) - n(A' \cap B')$

$= 50 - 3$

$= 47.$

ii) $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$n(A \cap B) = 35 + 22 - 47$

$= 10$

iii) $n(A' \cap B) = n(B) - n(A \cap B)$

$= 22 - 10$

$= 12$

iv) $n(A \cup B') = n(X) - n(A' \cap B)$

$= 50 - 12$

$= 38$

Ex.8: In an examination; 40 students failed in Physics, 40 in Chemistry and 35 in Mathematics; 20 failed in Mathematics and Physics, 17 in Physics and Chemistry, 15 in Mathematics and Chemistry and 5 in all the three subjects. If 350 students appeared for the examination, how many of them did not fail in any of the three subjects?

Solution:

P = set of students failed in Physics

C = Set of students failed in Chemistry

M = set of students failed in Mathematics

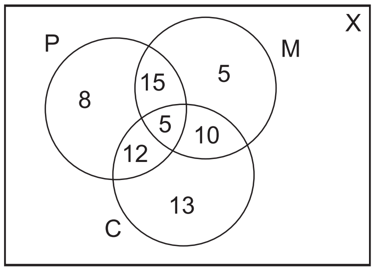


Fig. 1.19

From figure 1.19, we have

$n(X) = 350$, $n(P) = 40$, $n(C) = 40$, $n(M) = 35$

$n(M \cap P) = 20$, $n(P \cap C) = 17$, $n(M \cap C) = 15$

and $n(M \cap P \cap C) = 5$

The number of students who failed in at least one subject = $n(M \cup P \cup C)$

we have,

$n(M \cup P \cup C) = n(M) + n(P) + n(C) + n(M \cap P \cap C)$

$-n(M \cap P) - (P \cap C) - n(M \cap C)$

$= 35 + 40 + 40 + 5 - 20 - 17 - 15$

$= 68$

The number of students who did not fail in any subject

$= 350 - 68$

$= 282$

Ex. 9: A company produces three kinds of products A, B and C. The company studied the preference of 1600 consumers and found that the product A was liked by 1250, the product B was liked by 930 and product C was liked by 1000. The proudcts A and B were liked by 650, the products B and C were liked by 610 and the products C and A were liked by 700 consumers. 30 consumers did not like any of these three products

Find number of consumers who liked.

i) all the three products

ii) only two of these products.

Solution: Given that totally 1600 consumers were studied.

$\therefore n(X) = 1600$,

Let A be the set of all consumers who liked product A. Let B the set of all consumers who liked product B and C be the set of all consumers who liked product C.

$n(A) = 1250$,

$n(B) = 930$, $n(C) = 1000$.

$n(A\cap B) = 650$, $n(B\cap C) = 610$

$n(A\cap C) = 700$, $n(A'\cap B'\cap C') = 30$

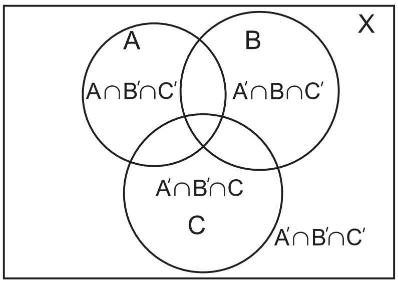


Fig. 1.20

i) $n(A\cup B\cup C) = n(X) - n[(A\cup B\cup C)]'$

$= n(X) - n(A'\cap B'\cap C')$

$= 1600 - 30$

$= 1570$

$\therefore\ n(A\cup B\cup C) = n(A) + n(B) + n(C)$

$- n(A\cap B) - n(B\cap C) - n(C\cap A)$

$+ n(A\cap B\cap C)$

$\therefore\ 1570 = 1250 + 930 + 1000 - 650 - 610$

$- 700 + n(A\cap B\cap C)$

$\therefore\ n(A\cap B\cap C) = 1570 + 1960 - 3180$

$= 350$

$\therefore$ The number of consumers who liked all the three products is 350.

ii) $n[(A\cap B)\cap C'] = n(A\cap B) - n[(A\cap B)\cap C]$

$= 650 - 350$

$= 300$

Similarly $n(A'\cap B\cap C) = 610 - 350 = 260$

and $n(A\cap B'\cap C) = 700 - 350 = 350$

$\therefore$ The number of consumers who liked only two of the three products are

$= n(A\cap B\cap C') + n(A\cap B'\cap C) + n(A'\cap B\cap C)$

$= 300 + 350 + 260$

$= 910$

EXERCISE 1.1

1) Describe the following sets in Roster form

i) $\{x/x$ is a letter of the word 'MARRIAGE'$\}$

ii) $\{x/x$ is an integer, $-\frac{1}{2} < x < \frac{9}{2}\}$

iii) $\{x/x = 2n, n\in N\}$

2) Describe the following sets in Set–Builder form

i) $\{0\}$

ii) $\{0, \pm1, \pm2, \pm3\}$

iii) $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$

3) If $A = \{x/6x^2+x-15 = 0\}$

$B = \{x/2x^2-5x-3 = 0\}$

$C = \{x/2x^2-x-3 = 0\}$ then

find i) $(A\cup B\cup C)$ ii) $(A\cap B\cap C)$

4) If A, B, C are the sets for the letters in the words 'college', 'marriage' and 'luggage' respectively, then verify that

$A-(B\cup C) = (A-B)\cap(A-C)$

5) If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {4, 5, 6, 7, 8} and universal set X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, then verify the following:

i) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

ii) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

iii) $(A\cup B)' = (A'\cap B')$

iv) $(A\cap B)' = A'\cup B'$

v) $A = (A\cap B)\cup(A\cap B')$

vi) $B = (A\cap B)\cup(A'\cap B)$

vii) $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

6) If A and B are subsets of the universal set X and $n(X) = 50$, $n(A) = 35$, $n(B) = 20$, $n(A'\cap B') = 5$, find i) $n(A\cup B)$ ii) $n(A\cap B)$ iii) $n(A'\cap B)$ iv) $n(A\cap B')$

7) Out of 200 students; 35 students failed in MHT–CET, 40 in AIEEE and 40 in IIT entrance, 20 failed in MHT–CET and AIEEE, 17 in AIEEE and IIT entrance, 15 in MHT–CET and IIT entrance and 5 failed in all three examinations. Find how many students.

i) did not fail in any examination.

ii) failed in AIEEE or IIT entrance.

8) From amongst 2000 literate individuals of a town, 70% read Marathi newspapers, 50% read English newspapers and 32.5% read both Marathi and English newspapers. Find the number of individuals who read.

i) at least one of the newspapers.

ii) neither Marathi nor English newspaper.

iii) Only one of the newspapers.

9) In a hostel, 25 students take tea, 20 students take coffee, 15 students take milk, 10 student take both tea and coffee, 8 students take both milk and coffee. None of them take tea and milk both and everyone takes atleast one beverage, find the number of students in the hostel.

10) There are 260 persons with a skin disorder. If 150 had been exposed to the chemical A, 74 to the chemical B, and 36 to both chemicals A and B, find the number of persons exposed to

i) Chemical A but not Chemical B

ii) Chemical B but not Chemical A

iii) Chemical A or Chemical B

11) If A = {1,2,3} write the set of all possible subsets of A.

12) Write the following intervals in set–builder form.

i) [–3, 0] ii) [6,12]

iii) (6, 12] iv) [–23, 5)

1.4 RELATIONS:

1.4.1 Ordered Pair:

A pair (a,b) of numbers, such that the order, in which the numbers appear is important, is called an ordered pair. In general, ordered pairs (a,b) and (b, a) are different. In ordered pair (a,b), 'a' is called the first component and 'b' is called the second component.

Two ordered pairs (a,b) and (c, d) are equal, if and only if $a = c$ and $b = d$.

Also, $(a, b) = (b, a)$ if and only if $a = b$

Ex. 1: Find x and y when $(x + 3, 2) = (4, y - 3)$

Solution: Using the definition of equality of two ordered pairs, we have

$(x + 3, 2) = (4, y - 3)$

$\therefore\ x + 3 = 4$ and $2 = y - 3$

$\therefore\ x = 1$ and $y = 5$

1.4.2 Cartesian Product of two sets:

Let A and B be two non–empty sets then the cartesian product of A and B is defined as the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. It is denoted as A×B and read as 'A cross B'

Thus, $A\times B = \{(a, b) / a\in A, b\in B\}$

For example,

If $A = \{1, 2\}$ and $B = \{a, b, c\}$ then

$A\times B = \{(1, a) (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Let's Note.

If $A = \phi$ or $B = \phi$, then

$\therefore\ A\times B = \phi$

1.4.3 Number of elements in the Cartesian product of two finite sets:

Let A and B be any two finite sets with $n(A) = m_1$, and $n(B) = m_2$, then the number of elements in the Cartesian product of A and B is given by

$$n(A\times B) = m_1 \bullet m_2 = n(A) \bullet n(B)$$

Ex. 1) : Let $A = \{1, 3\}$; $B = \{2, 3, 4\}$ Find the number of elements in the Cartesian product of A and B.

Solution: Given $A = \{1, 3\}$ and $B = \{2, 3, 4\}$

$\therefore\quad n(A) = 2$ and $n(B) = 3$

$\therefore\quad n(A\times B) = 2\times 3 = 6$

1.4.4 Relation (Definition): If A and B are two non empty sets then any subset of A×B is called relation from A to B and is denoted by capital letters P, Q, R etc..

If R is a relation and $(x, y) \in R$ then it is denoted by xRy, read as x is related to y under the relation R or $R : x \to y$.

y is called 'image' of x under R and x is called 'pre–image' of y under R.

Ex. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 9\}$

Let R be a relation such that $(x, y) \in R$ if $x^2 = y$. List the elements of R.

Solution: Here $A = \{1, 2, 3, 4, 5\}$

and $B = \{1, 4, 9\}$

$\therefore\ R = \{(1, 1), (2, 4), (3, 9)\}$

Arrow diagram fot this relation R is given by

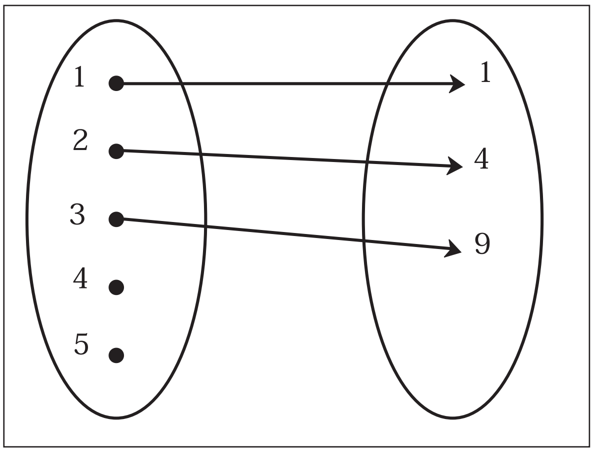


Fig. 1.21

1) Domain:

The set of all first components of the ordered pairs in a relation R is called the domain of the relation R.

i.e. domain $(R) = \{a \in A \mid (a, b) \in R\}$

2) Co-domain:

If R is a relation from A to B then set B is called the co–domain of the relation R

3) Range:

The set of all second components of all ordered pairs in a relation R is called the range of the relation.

i.e. Range $(R) = \{b\in B \mid (a, b) \in R\}$

1.5 TYPES OF RELATIONS:

i) One-One Relation(Injective): A relation R from A to B is said to be one–one if every element of A has at most one image in B under R and distinct elements in A have distinct images in B under R.

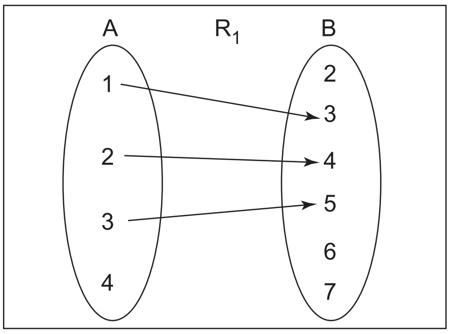


Fig. 1.22

For example: Let A= {1,2,3,4}

B = {2,3,4,5,6,7}

and R_1 = {(1, 3), (2, 4), (3, 5)}

Then R_1 is a one – one relation. (fig 1.22)

Here, domain of R_1 = {1, 2, 3} and range is {3, 4, 5}

ii) Many-one relation: A relation R from A to B is said to be many – one if two or more than two elements in A have same image in B.

For example: Let R_2 = {(1, 4), (3, 7), (4, 4)}

Then R_2 is many – one relation from A to B

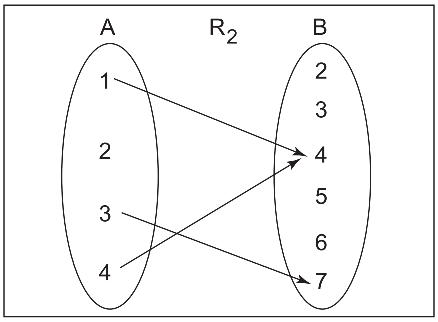


Fig. 1.23

Domain of R_2 = {1, 3, 4}

Range of R_2 = {4, 7}

iii) Into relation: A relation R from A to B is said to be into relation if there exists at least one element in B which has no pre–image in A. i.e. Range of R is proper subset of B

For example: Let A = {–2, –1, 0, 1, 2, 3}

B = {0, 1, 2, 3, 4}

Let R_3 = {(–2, 4), (–1, 1), (0, 0), (1, 1), (2, 4)}

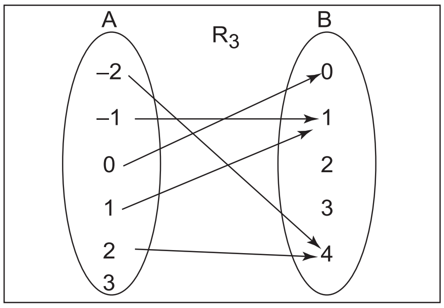


Fig. 1.24

∴ Range of R_3 = {0, 1, 4}

∴ Range of $R_3 \subset$ B

Then R_3 is an into relation from A into B

iv) Onto relation (Surjective) : A relation R from A to B is said to be onto relation, if every element of B is the image of some element of A.

For Example: Let A = {–3 , –2, –1, 1, 3, 4}

B = {1, 4, 9}. Let

R_4 = {(–3, 9), (–2, 4), (–1, 1), (1, 1), (3, 9)}

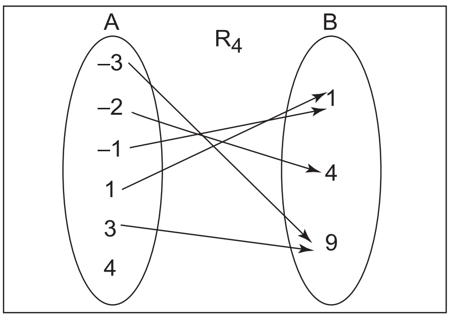


Fig. 1.25

$\therefore$ Range or R_4 = {1, 4, 9}

$\therefore$ Range = co–domain (B)

Thus R_4 is an onto relation from A to B.

Binary relation on a set:

Let A be non–empty set then every subset of A× A is called a binary relation on A.

Let's Note.

A relation having the same set as domain and co-domain is a binary relation on that set.

SOLVED EXAMPLES

Ex. 1.: Let A = {1, 2, 3} and

R = {(1, 2), (2, 2), (3, 1), (3, 2)}

AxA = {(1, 1), (1, 2), 1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Clearly R⊂A×A and therefore, R is a binary relation on A.

Ex. 2: Let N be the set of all natural numbers and R = {*a*, *b*) / *a*, *b* ∈ N and 2*a* + *b* = 10}

Since R⊂N × N, R is binary relation on N. Clearly, R = {(1, 8), (2, 6), (3, 4), (4, 2)}

We can state domain, range and co–domain of the relation R as follows :

Domain (R) = {1, 2, 3, 4}

Range (R) = {2, 4, 6, 8}

Co–domain = N.

i) $\phi \subset$ A× A is a relation on A and is called the empty or void relation on A.

ii) A× A $\subseteq$ A× A. So it is a relation on A called the universal relation i.e. R = A× A

Ex. 3: If A = {2, 4, 6}

then R = A× A = {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}

and R = A× A is the universal relation on A.

Note: The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A× B

If n(A) = m_1 and n(B) = m_2

then n(A× B) = m_1m_2

and the total number of relations is $2^{m_1 \cdot m_2}$

Properties of relations:

Let A be a non–empty set. Then a relation R on A is said to be

(i) Reflexive, if (*a*, *a*) ∈ R for every *a*∈A i.e. *aRa* for every *a*∈A

(ii) Symmetric, if (*a*, *b*)∈R
⇒ (*b*, *a*) ∈R for all *a*, *b*, ∈A
i.e. aRb ⇒ bRa for all *a*, *b*∈A

(iii) Transitive, if (*a*, *b*)∈R and (*b*, *c*) ∈ R
⇒ (*a*, *c*)∈R for all *a*, *b*, *c*∈A.

Note: Read the symbol " ⇒" as "implies".

Equivalence relation:

A relation which is reflexive, symmetric and transitive is called an equivalence relation.

SOLVED EXAMPLES

Ex. 1: Let R be a relation on Q, defined by
R = {(*a*, *b*)/*a*, *b*∈Q and *a*–*b*∈Z}

Show that R is an equivalence relation.

Solution: Given

R = {(*a*, *b*)/*a*, *b*∈Q and *a*–*b*∈Z}

i) Let a ∈Q then *a* – *a* = 0∈Z

$\therefore$ (*a*, *a*)∈R

So, R is reflexive.

ii) $(a, b) \in R \Rightarrow (a-b) \in Z$

i.e. $(a-b)$ is an integer

$\Rightarrow -(a-b)$ is an integer

$\Rightarrow (b-a)$ is an integer

$\Rightarrow (b, a) \in R$

Thus $(a, b) \in R \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

iii) $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a-b)$ is an integer and $(b-c)$ is an integer

$\Rightarrow \{(a-b) + (b-c)\}$ is an integer

$\Rightarrow (a-c)$ is an integer

$\Rightarrow (a, c) \in R$

Thus $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

SOLVED EXAMPLES

Ex. 1: If $(x+1, y-2) = (3, 1)$ find the value of x and y.

Solution: Since the order pairs are equal, the corresponding elements are equal.

$\therefore x + 1 = 3$ and $y-2 = 1$

$\therefore x = 2$ and $y = 3$

Ex. 2: If A = {1, 2}, find A×A

Solution: We have A = {1, 2}

$\therefore$ A×A = {(1, 1), (1, 2), (2, 1), (2, 2)}

Ex. 3: If A = {1, 3, 5} and B = {2, 3}. Find A×B and B×A. Is A×B = B×A?

Solution: We have

A×B = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)} and B×A = {(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)} All elements in A×B, B×A (except (3, 3) are different.

$\therefore A\times B \neq B\times A$

Ex. 4: If A×B = {(3, 2), (3, 4), (5, 2), (5, 4)} then find A and B

Solution: Clearly, we have

A = Set of all first components of A×B

$\therefore$ A = {3, 5}

B = Set of all second components of A×B

$\therefore$ B = {2, 4}

Thus A = {3, 5} and B = {2, 4}

Ex. 5: Express $\{(x, y)/x^2 + y^2 = 25$ where $x, y \in W\}$ as a set of ordered pairs.

Solution: We have $x^2 + y^2 = 25$

$\therefore x = 0, y = 5 \Rightarrow x^2 + y^2 = 0^2 + 5^2 = 25$

$x = 3, y = 4 \Rightarrow x^2 + y^2 = (3)^2 + (4)^2 = 25$

$x = 4, y = 3 \Rightarrow x^2 + y^2 = (4)^2 + (3)^2 = 25$

$x = 5, y = 0 \Rightarrow x^2 + y^2 = (5)^2 + (0)^2 = 25$

$\therefore$ The given set = {(0, 5), (3, 4), (4, 3), (5, 0)}

Ex. 6: Let A = {1, 2, 3} and B = {2, 4, 6}

Show that R = {(1, 2), (1, 4), (3, 2), (3, 4)} is a relation from A to B find

i) domain (R) ii) Co–domain (R)

iii) Range (R)

Also represent this relation by arrow diagram

Solution: Here A = {1, 2, 3}, B = {2, 4, 6}

and R = {(1, 2), (1, 4), (3, 2), (3, 4)}

Since $R \subset A\times B$, R is a relation from A to B

i) Domain (R) = Set of first components of R = {1, 3}

ii) Co–domain (R) = B = {2, 4, 6}

iii) Range (R) = Set of second components of R = {2, 4}

Arrow Diagram:

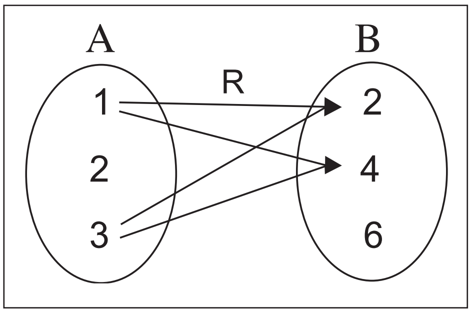


Fig. 1.26

Ex. 7 : Let A = {1, 2, 3, 4, 5} and B = {1, 4, 5}

Let R be a relation from A to B such that

$(x, y) \in R$ if $x < y$

i) List the elements of R.

ii) Find the domain, co–domain and range of R.

iii) Draw the above relation by an arrow diagram.

Solution: A = {1, 2, 3, 4, 5} and B = {1, 4, 5}

i) The elements of R are as follow:

R = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5) (4, 5)}

ii) Domain (R) = {1, 2, 3, 4}

Range (R) = {4, 5}

Co–domain (R) = {1, 4, 5} = B

iii) An Arrow diagram

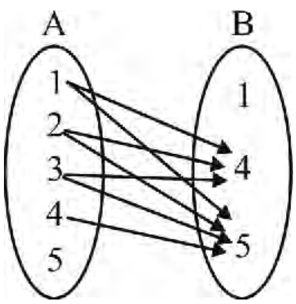


Fig. 1.27

Ex. 8 : Let A = {1, 2, 3, 4, 5, 6 } define a relation R from A to A by R = $\{(x, y) / y = x + 1\}$

i) Draw this relation using an arrow diagram.

ii) Write down the domain, co-domain and range or R.

Solution :

1) A relation R from A to A by

R = $\{(x,y) / y=x+1\}$ is given by

R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

The corresponding arrow diagram is

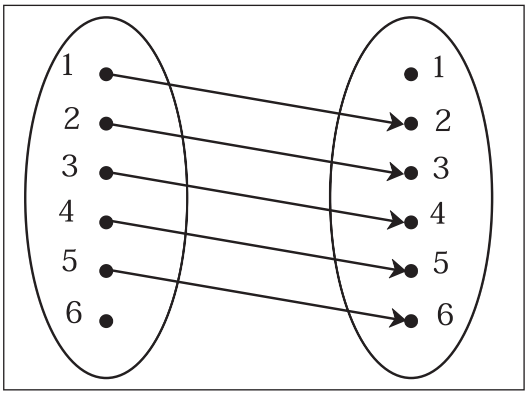


Fig. 1.28

ii) Domain = {1, 2, 3, 4, 5}

Range = {2 , 3, 4, 5, 6}

Co-domain = { 1, 2, 3, 4, 5, 6}

EXERCISE 1.2

1) If $(x - 1, y + 4) = (1, 2)$ find the values of x and y.

2) If $\left(x+\frac{1}{3}, \frac{y}{3}-1\right) = \left(\frac{1}{3}, \frac{3}{2}\right)$, find x and y.

3) If A = $\{a, b, c\}$, B = $(x , y\}$ find

A × B, B× A, A × A, B× B.

4) If P = { 1, 2, 3} and Q = {6, 4},

find the sets $P \times Q$ and $Q \times P$

5) Let A = { 1, 2, 3, 4}, B = { 4, 5, 6}, C = { 5, 6}.

Find i) $A \times (B \cap C)$

ii) $(A \times B) \cap (A \times C)$

iii) $A \times (B \cup C)$

iv) $(A \times B) \cup (A \times C)$

6) Express $\{(x, y) / x^2 + y^2 = 100$ where $x, y \in W\}$ as a set of ordered pairs.

7) Write the domain and range of the following relations.

i) $\{(a, b) / a \in N, a<6$ and $b = 4\}$

ii) $\{(a, b) / a, b \in N, a+b = 12\}$

iii)(2, 4), (2, 5), (2,6), (2, 7)}

8) Let A = {6, 8} and B = {1, 3, 5}

Let R = $\{(a, b)/a\in$ A, $b\in$B, a-b is an even number}

Show that R is an empty relation from A to B.

9) Write the relation in the Roster form and hence find its domain and range.

i) $R_1 = \{(a, a^2)$ / a is prime number less than 15}

ii) $R_2 = \{(a, \frac{1}{a}) / 0<a \le 5, a\in N\}$

10) R = {(a, b) / b = a + 1, a∈z, 0<a<5}

Find the Range of R

11) Find the following relation as sets of ordered pairs.

i) $\{(x, y) / y = 3x, x\in \{1,2,3\}, y\in \{3,6,9,12\}\}$

ii) $\{(x, y) / y > x + 1, x \in \{1, 2\}$ and $y \in \{2, 4, 6\}\}$

iii) $\{(x, y) / x+y = 3, x, y\in \{0, 1, 2, 3\}\}$

Let's remember!

- A set is a collection of well defined objects.
- A set which does not contain any element is called an empty set and is denoted by ϕ.
- The power set of a set A is the sets of all subsets of A and is denoted by P(A).
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- For any sets A and B
  $(A \cup B)' = A' \cap B'$
- If A and B are finite sets such that : $A \cap B = \phi$, then
  $n(A \cup B) = n(A) + n(B)$
- If $n(A \cap B) \neq \phi$, then
  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $A \times B = \{(a, b)/a\in A, b\in B\}$
  In particular, $R \times R = \{(x, y) / x, y \in R\}$
- If $(a, b) = (x, y)$ then $a = x$ and $b = y$
- If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$
- $A \times \phi = \phi$
- The image of an element x under a relation R is given by y, where $(x, y) \in$ R. (relation).
- The domain of R is the set of all 1st components of the ordered pairs in R (relation).
- The range of R (relation) is the set of all second components of the ordered pairs in R
- Let A be any non-empty set, then every subset of $A \times A$ is binary relation on A.

MISCELLANEOUS EXERCISE - 1

1) Write the following sets in set builder form

i) {10, 20, 30, 40, 50},

ii) { a, e, i, o, u)

iii) {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

2) If U = {x/x∈N, $1 \le x \le 12$}

A = { 1, 4, 7, 10}   B = {2, 4, 6, 7, 11}

C = {3, 5, 8, 9, 12}

Write the sets

i) $A \cup B$ ii) $B \cap C$ iii) A-B

iv) B-C   v) $A \cup B \cup C$ vi) $A \cap (B \cup C)$

3) In a survey of 425 students in a school, it was found that 115 drink apple juice, 160 drink orange juice and 80 drink both apple as well as orange juice. How many drink neither apple juice nor orange juice?

4) In a school there are 20 teachers who teach Mathematics or Physics, of these, 12 teach Mathematics and 4 teach both Physics and Mathematics. How many teachers teach Physics?

5) i) If A = {1, 2, 3} and B = {2, 4}, state the elements of $A \times A$, $A \times B$, $B \times A$, $B \times B$, $(A \times B) \cap (B \times A)$

ii) If A = {-1, 1} , find $A \times A \times A$

6) If A = {1, 2, 3}, B = {4, 5, 6} which of following are relations from A to B

i) R_1 = {(1, 4), (1, 5), (1, 6)}

ii) R_2 = {(1, 5), (2, 4), (3, 6)}

iii) R_3 = {(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)}

iv) R_4 = {(4, 2), (2, 6), (5, 1), (2, 4)}

7) Determine the Domain and range of the following relations.

R = { (a, b) / a∈N, a<5, b = 4}

ACTIVITIES

Activity 1.1 :

Take Universal set X having 10 elements and take two unequal subsets A and B of set X. Write A', B', A – B, B – A, $A \cup B$, $A \cap B$, $(A \cup B)'$, $(A \cap B)'$.

Activity 1.2 :

Give an example of nonempty sets A and B and universal set such that

i) $A \cup B = A \cap B$ ii) $(A \cup B)' = A' \cup B'$

iii) $A' \cap B' = (A \cap B)'$

Activity 1.3 :

By taking suitable example verify that

$A \times B \neq B \times A$. But $n(A \times B) = n(B \times A)$.

Activity 1.4 :

What conclusion will you draw about two sets A and B if $A \subseteq B$ and $B \subseteq A$.

Activity 1.5 :

If A = $\left\{ \frac{-1}{2}, \frac{2}{3} \right\}$ then it can also be expressed as

A = {x/ $\boxed{\cdots}\, x^2 + \boxed{\cdots}\, x + \boxed{\cdots} = 0$}.

Activity 1.6 :

Write the domain and range for the following relation.

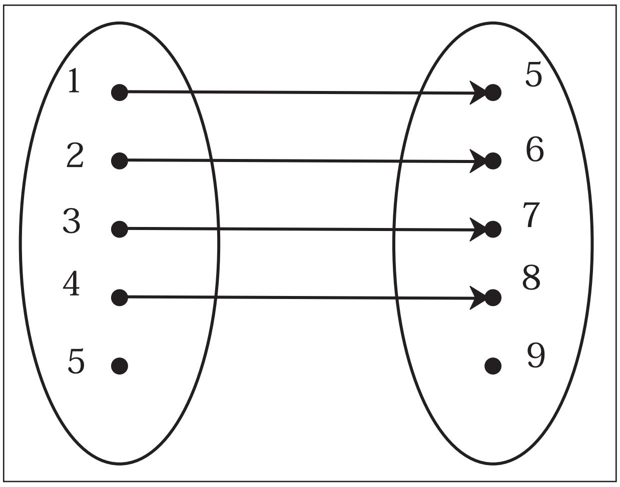


Fig.1.29

1) Write the relation in term of the ordered pairs.

2) Write the image of 5. (if it exists)

3) Write the pre-image of 8.

4) Write the relation in terms of the formula.

Activity 1.7 :

Write the following sets in Roster form. Also draw Venn diagram.

i) A - set of all factors of 24.

ii) B - set of all prime numbers less than 30.

iii) C - Set of all letters in the word 'MATHEMATICS'.

Activity 1.8 :

In a survey of 400 students. It was found that 150 drink milk, 250 drink Tea, 50 drink both. How many drink neither Tea nor milk. Solve using formula & Venn diagram.

Let M is set of students who drink milk.

Let T is set of students who drink Tea.

n(M) = ☐

n(T) = ☐

$n(M \cup T) = n(M) + n(T) - n(M \cap T)$

= ☐ + ☐ – ☐

= ☐

Number of students neither drink Tea nor milk = Total number of students – $n(M \cup T)$

= ☐ – ☐ = ☐

Activity 1.9 :

Complete the following activity.

$A = \{\frac{1}{3x} / x \in N \ \& \ x < 8\}$

$B = \{\frac{1}{2x} / x \in N \ \& \ x \leq 8\}$

Find $A \cup B$, $A \cap B$, A – B, B – A

Solution :

Write set A & set B in list form

A = {..................}

B = {..................}

For $A \cup B$, [consider all elements from A as well as B, don't repeat elements]

$\therefore$ $A \cup B$ = {....................}

For $A \cap B$, [Take all the elements that are common in A and B]

$\therefore$ $A \cap B$ = {......................}

For A – B [Take all the elements that are present in A but not in B]

$\therefore$ A – B = {..................}

B – A = {..................}

Activity 1.10 :

U = {1,2,3,4,5,6,7,8}

A = {1,2,3,4,,5} , A' = {................}

B = {4,5,6,7,8} , B' = {................}

Complete the following activity.

$A \cup B$ = {.......................}, $n(A \cup B)$ = ☐

$A - B$ = {.......................}, $n(A - B)$ = ☐

$B - A$ = {.......................}, $n(B - A)$ = ☐

$A \cap B$ = {.......................}, $n(A \cap B)$ = ☐

i) $n(A - B) + n(A \cap B) + n(B - A)$

= ☐ + ☐ + ☐

= ☐

ii) $A \cap B'$ = {.......................}

$A \cap B' = A - B$

iii) $A' \cap B$ = {.......................}

$A' \cap B = B - A$

❖ ❖ ❖

2. FUNCTIONS

Let's study.

- Function, Domain, Co-domain, Range
- Types of functions – One-one, Onto
- Representation of function
- Evaluation of function
- Fundamental types of functions
- Some special functions

Let's learn.

2.1 FUNCTION

Definition : A function f from set A to set B is a relation which associates each element x in A, to a unique (exactly one) element y in B.

Then the element y is expressed as $y = f(x)$.

y is the image of x under f

If such a function exists, then A is called the **domain** of f and B is called the **co-domain** of f.

For example,

Examine the following relations which are given by arrows of line segments joining elements in A and elements in B.

1)

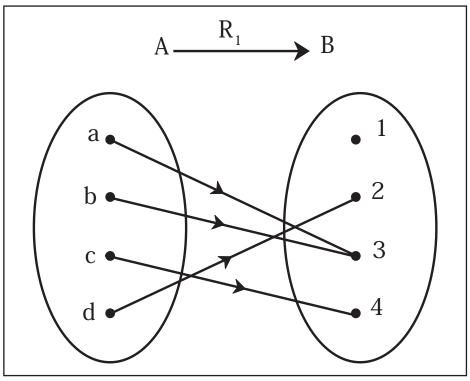


Fig. 2.1

R_1 is a well defined function.

2)

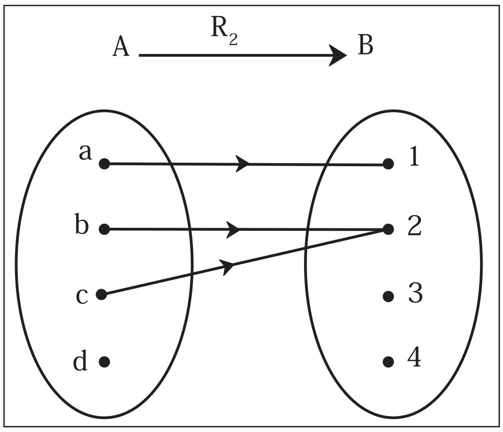


Fig. 2.2

R_2 is not a function because element d in A is not associated to any element in B.

3)

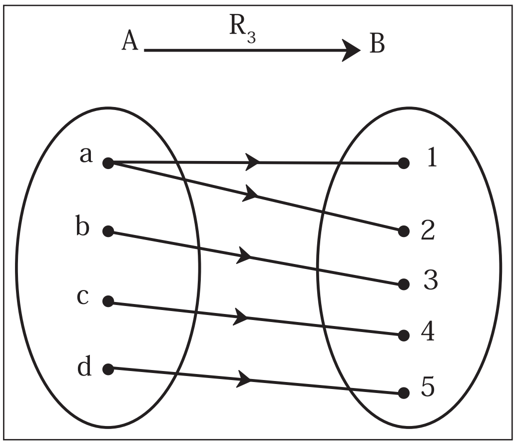


Fig. 2.3

R_3 is not a function because element a in A is associated to two elements in B.

The relation which defines a function f from domain A to co-domain B is often given by an algebraic rule.

2.1.1 Types of function

One-one or One to one or Injective function

Definition : A function $f : A \rightarrow B$ is said to be one-one if distinct elements in A have distinct images in B. The condition is also expressed as

$a \neq b \Rightarrow f(a) \neq f(b)$

Onto or **Surjective function**

Definition: A function $f : A \rightarrow B$ is said to be onto if every element y in B is an image of some x in A

The image of A can be denoted by f (A).

$f(A) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$

f (A) is also called the **range** of A.

Note that $f : A \rightarrow B$ is onto if $f(A) = B$.

For example,

1)

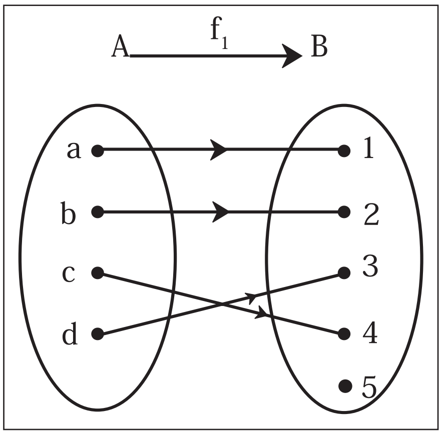


Fig. 2.4

f_1 is one-one, not onto

2)

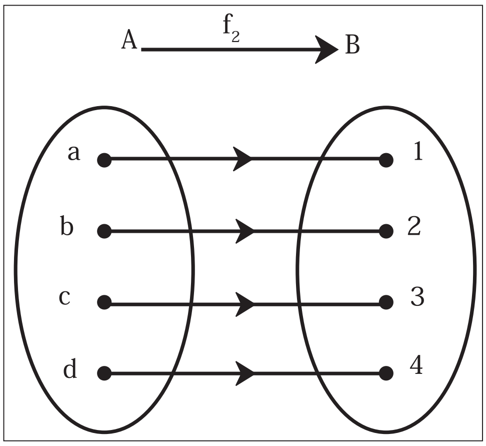


Fig. 2.5

f_2 is one-one, onto

3)

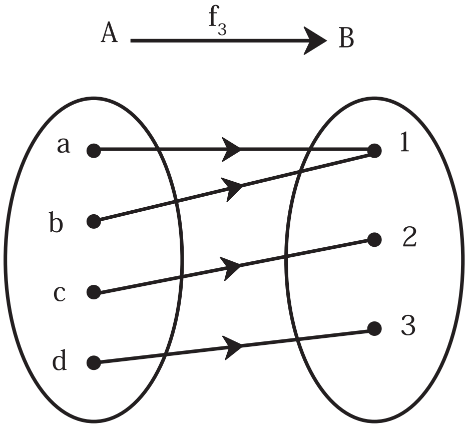


Fig. 2.6

f_3 is not one-one, onto

4)

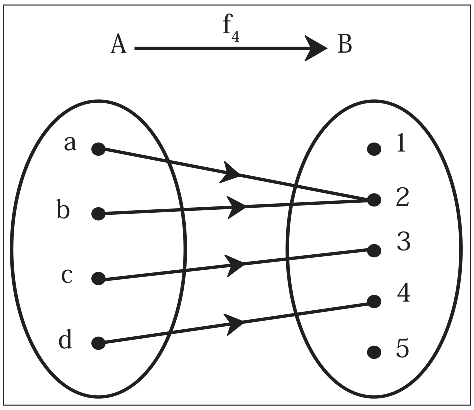
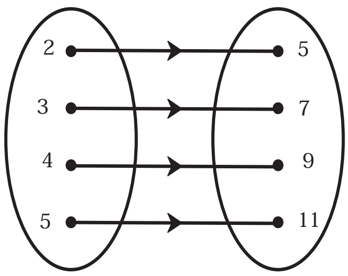
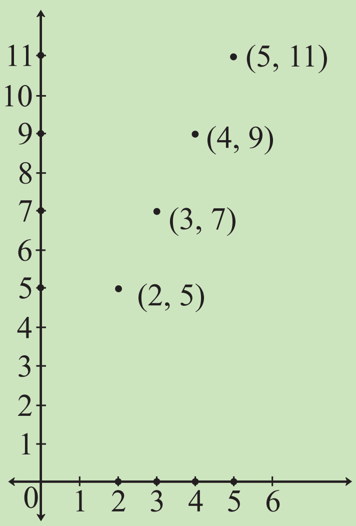


Fig. 2.7

f_4 is not one-one , not onto

Representation of Function

Verbal form	Output is 1 more than twice the input Domain : Set of inputs Range : Set of outputs
Arrow form/ Venn Diagram Form	 **Fig. 2.7(A)** Domain : Set of pre-images Range: Set of images

<table>
<tr><td>Ordered Pair
(x, y)</td><td>$f = \{(2,5), (3,7), (4,9), (5,11)\}$
Domain : Set of 1st components from ordered pair = $\{2, 3, 4, 5\}$
Range : Set of 2nd components from ordered pair = $\{5, 7, 9, 11\}$</td></tr>
<tr><td>Rule / Formula</td><td>$y = f(x) = 2x + 1$
Where $x \in N$, $1 < x < 6$
$f(x)$ read as 'f of x' or 'function of x'
Domain : Set of values of x,
Range : Set of values of y for which $f(x)$ is defined</td></tr>
<tr><td>Tabular Form</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr><tr><td>5</td><td>11</td></tr></table>Domain : x values
Range: y values</td></tr>
<tr><td>Graphical form</td><td>
Fig. 2.7(B)
Domain: Extent of graph on x-axis.
Range: Extent of graph on y-axis.</td></tr>
</table>

2.1.2 Graph of a function:

If the domain of function is R , we can show the function by a graph in xy plane. The graph consists of points (x,y), where $y = f(x)$.

Evaluation of a function:

1) **Ex:** Evaluate $f(x) = 2x^2 - 3x + 4$ at

$x = 7$ and $x = -2t$

Solution : $f(x)$ at $x = 7$ is $f(7)$

$$f(7) = 2(7)^2 - 3(7) + 4$$
$$= 2(49) - 21 + 4$$
$$= 98 - 21 + 4$$
$$= 81$$
$$f(-2t) = 2(-2t)^2 - 3(-2t) + 4$$
$$= 2(4t^2) + 6t + 4$$
$$= 8t^2 + 6t + 4$$

2) Using the graph of $y = g(x)$, find $g(-4)$ and g (3)

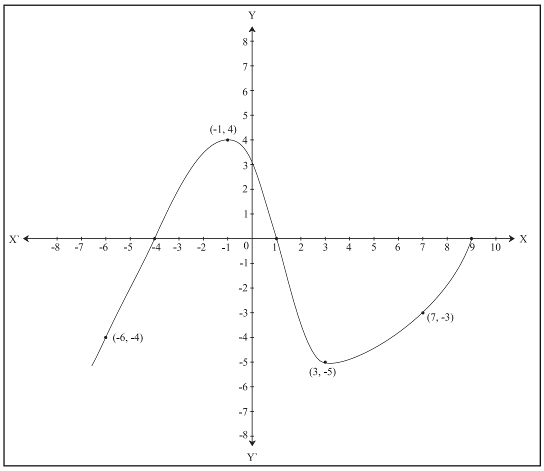


Fig. 2.8

From graph when $x = -4$, $y = 0$ so $g(-4) = 0$

From graph when $x = 3$, $y = -5$ so $g(3) = -5$

3) If $f(x) = 3x^2 - x$ and $f(m) = 4$, then find m

Solution : As

$$f(m) = 4$$
$$3m^2 - m = 4$$
$$3m^2 - m - 4 = 0$$
$$3m^2 - 4m + 3m - 4 = 0$$
$$m(3m - 4) + 1(3m - 4) = 0$$

$(3m - 4)(m + 1) = 0$

Therefore , $m = \frac{4}{3}$ or $m = -1$

4) From the graph below find x for which $f(x) = 4$

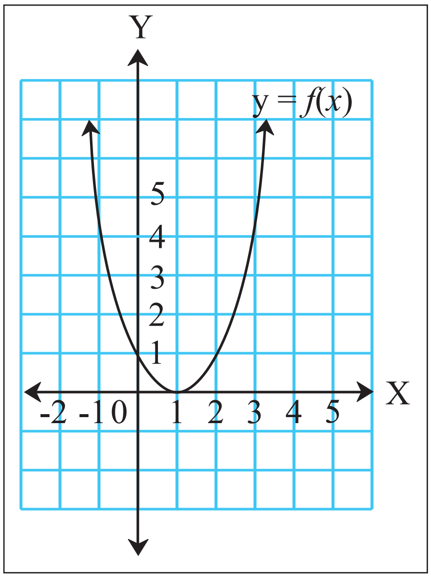


Fig. 2.9

Solution : To solve $f(x) = 4$

find the values of x where graph intersects line $y = 4$

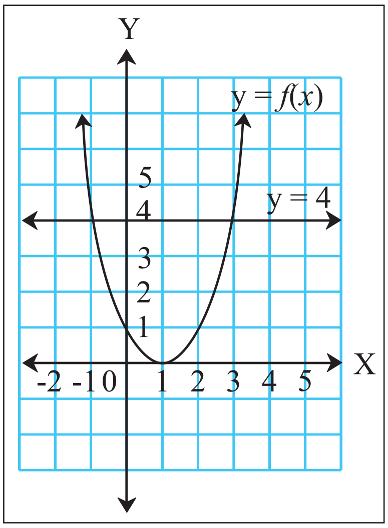


Fig. 2.10

Therefore , $x = -1$ and $x = 3$ are the values of x for which $f(x) = 4$

5) From the equation $4x + 7y = 1$ express

i) y as a function of x

ii) x as a function of y

Solution : Given relation is $4x + 7y = 1$

i) From the given equation

$7y = 1 - 4x$

$y = \frac{1-4x}{7}$ = function of x

So $y = f(x) = \frac{1-4x}{7}$

ii) From the given equation

$4x = 1 - 7y$

$x = \frac{1-7y}{4}$ = function of y

So $x = g(y) = \frac{1-7y}{4}$

Some Fundamental Functions

1) Constant Function :

$f : R \rightarrow R$ given by

$f(x) = k$, $x \in R$, k is constant is called the constant function.

Example: $f(x) = 3$, $x \in R$

Graph of $f(x) = 3$

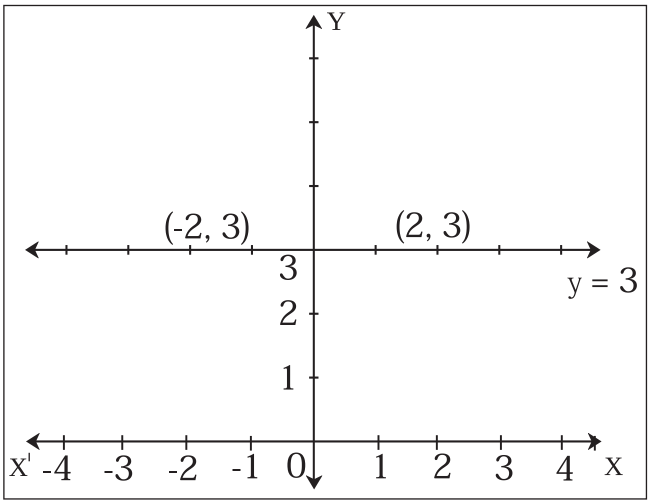


Fig. 2.11

Domain : R or $(-\infty, \infty)$ and **Range :** {3}

2) Identity function

If the domain and co-domain are same as A, then we define a special function as identity function defined by $f(x) = x$, for every $\mathbf{x} \in A$.

Let A=R. Then identity function $f : R \rightarrow R$, where $y = x$ is shown in the graph in fig. 2.12.

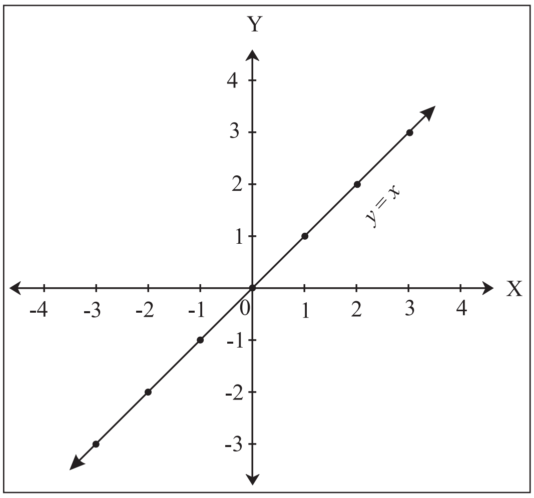


Fig. 2.12

Graph of $f(x) = x$

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

3) Linear Function :

Example : A function $f : R \rightarrow R$ of the form $f(x) = ax + b$

For example, $f(x) = -2x+3$, the graph of which is shown in Fig. 2.13

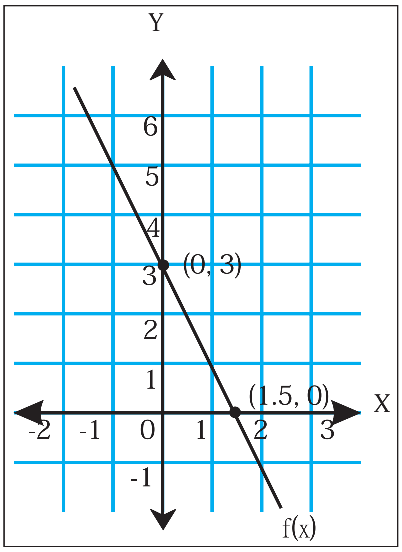


Fig. 2.13

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

Properties :

1) Graph of $f(x) = ax + b$ is a line with slope 'a', y-intercept 'b' and x-intercept $\left(-\frac{b}{a}\right)$.

2) Graph is increasing when slope is positive and decreasing when slope is negative.

4) Quadratic Function

Function of the form $f(x) = ax^2 + bx + c \ (a \neq 0)$

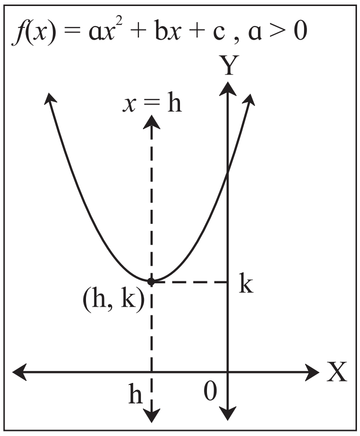


Fig. 2.14

Domain : R or $(-\infty, \infty)$ and **Range :** $[k, \infty)$

5) Function of the form $f(x) = ax^n$, $n \in N$

(Note that this function is a multiple of power of x)

i) Square Function

Example : $f(x) = x^2$

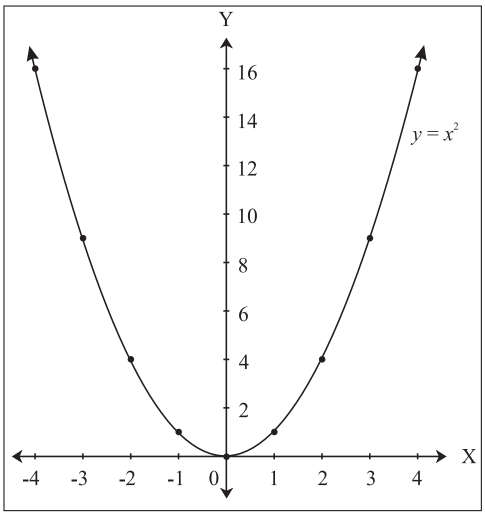


Fig. 2.15

Domain : R or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

ii) Cube Function

Example : $f(x) = x^3$

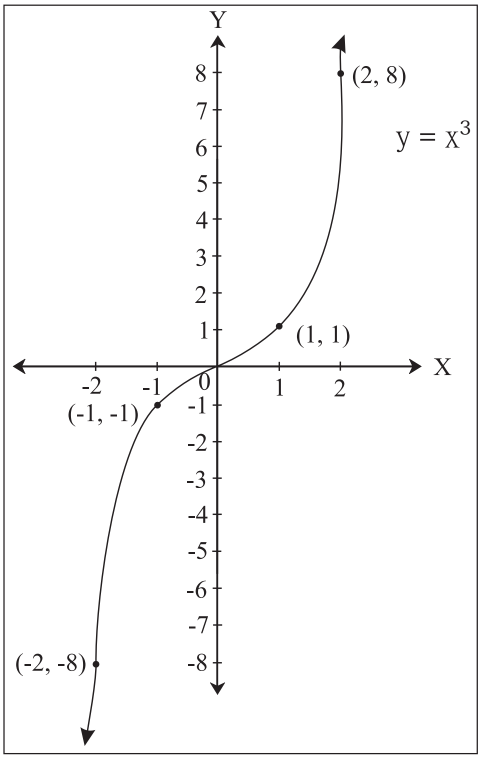


Fig. 2.16

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

6) Polynomial Function : A functions of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

is polynomial function of degree n, if $a_0 \neq 0$, and $a_0, a_1, a_2, \ldots\ldots, a_n$ are real. The graph of a general polynomial is more complicated and depends upon its individual terms.

Graph of $f(x) = x^3 - 1$

$f(x) = (x-1)(x^2+x+1)$ cuts x-axis at only one point (1,0) , which means $f(x)$ has one real root and two complex roots.

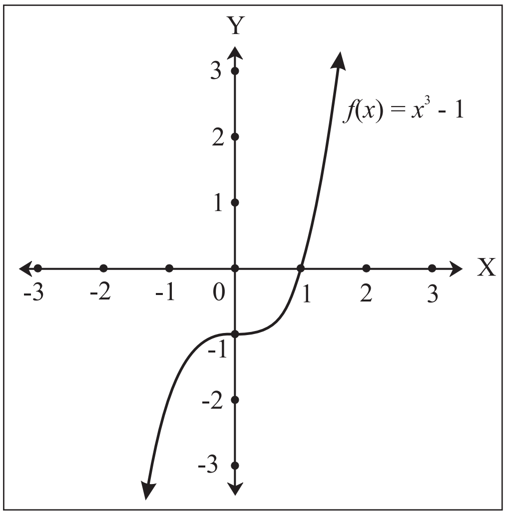


Fig. 2.17

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

7) Rational Function

Definition: Given polynomials $p(x)$, $q(x)$

$f(x) = \dfrac{p(x)}{q(x)}$ $(q(x) \neq 0)$ is called a rational function.

For example, $f(x) = \dfrac{1}{x}$

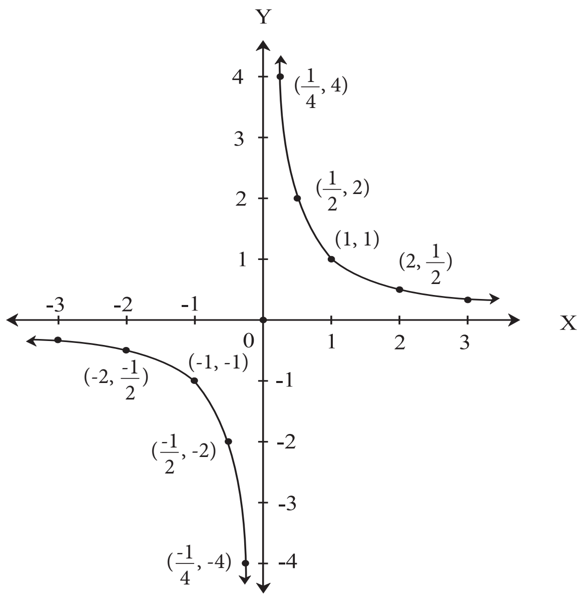


Fig. 2.18

Domain : R-{0} and **Range :** $R - \{0\}$

Example : Domain of function $f(x) = \frac{2x+9}{x^3-25x}$

i.e. $f(x) = \frac{2x+9}{x(x-5)(x+5)}$ is $R - \{0, 5, -5\}$ as for

$x = 0$, $x = -5$ and $x = 5$, denominator becomes 0 .

8) Exponential Function : A function $f : R \to R^+$ given by $f(x) = a^x$ is an exponential function with base a and exponent (or index) x, $a \neq 1$, $a > 0$ and $x \in R$.

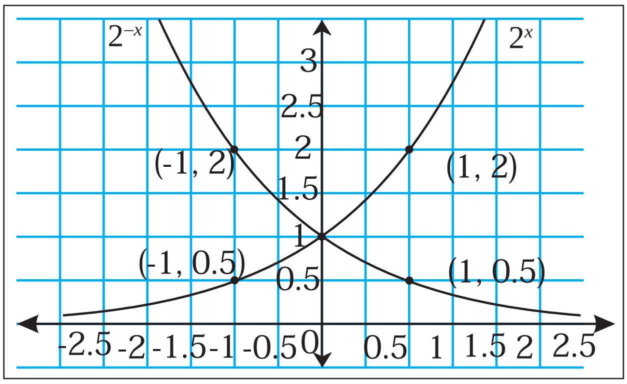


Fig. 2.19

Domain: R and **Range :** $(0, \infty) = R^+$

9) Logarithmic Function: A function f: $R^+ \to R$, defined as $f(x) = \log_a x$, ($a >$ o, $a \neq 1$, $x >$ o) where $y = \log_a x$ if and only if $x = a^y$ is called Logarithmic Function.

$$\underbrace{y = \log_a x}_{\textit{logarithmic form}} \text{ is equivalent to } \underbrace{a^y = x}_{\textit{exponential form}} .$$

Let's Note.

1) $y = \log_a x \Leftrightarrow a^y = x \Leftrightarrow$ anti $\log_a y = x$

2) As $a^0 = 1$, so $\log_a 1 = 0$ and as $a^1 = a$, so $\log_a a = 1$

3) As $a^x = a^y \Leftrightarrow x = y$ so $\log_a x = \log_a y \Leftrightarrow x = y$

4) For natural base e , $\log_e x = ln\, x$ $(x > 0)$ is called as Natural Logarithm Function.

Domain : $(0, \infty)$

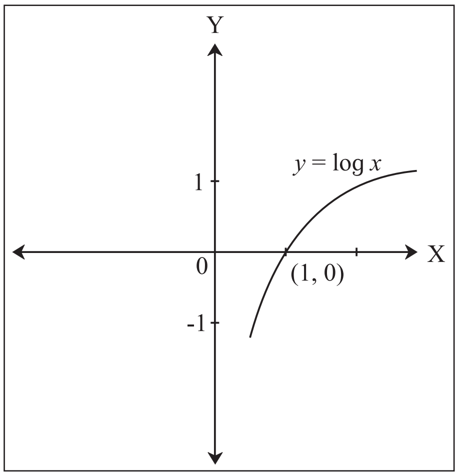


Fig. 2.20

Range : $(-\infty, \infty)$

Rules of Logarithm :

1) $\log_m ab = \log_m a + \log_m b$

2) $\log_m \frac{a}{b} = \log_m a - \log_m b$

3) $\log_m a^b = b.\log_m a$

4) Change of base formula:

For $a, x, b > 0$ and $a, b \neq 1$, $\log_a x = \frac{\log_b x}{\log_b a}$

Note: $\log_a x = \frac{\log_x x}{\log_x a} = \frac{1}{\log_x a}$

Algebra of functions:

If f and g are functions from X $\to$ R, then the functions $f + g$, $f - g$, fg, $\frac{f}{g}$ are defined as follows.

Operations
$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(f.g)(x) = f(x).g(x)$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Ex. If $f(x) = x^2 + 2$ and $g(x) = 5x - 8$, then find

i) $(f+g)(1)$

ii) $(f-g)(-2)$

ii) $(f.g)(3m)$

iv) $\frac{f}{g}(0)$

i) As $(f+g)(x) = f(x) + g(x)$

$(f+g)(1) = f(1) + g(1)$

$= (1)^2 + 2 + 5(1) - 8$

$= 1 + 2 + 5 - 8$

$= 8 - 8$

$= 0$

ii) As $(f-g)(x) = f(x) - g(x)$

$(f-g)(-2) = f(-2) - g(-2)$

$= [(-2)^2 + 2] + [5(-2) - 8]$

$= [4 + 2] - [-10-8]$

$= 6 + 18$

$= 24$

iii) As $(fg)(x) = f(x)\, g(x)$

$(f.g)(3m) = f(3m)\ \ g(3m)$

$= [(3m)^2 + 2]\ \ [5(3m) - 8]$

$= [9m^2 + 2]\ \ [15m-8]$

$= 135m^3 - 72m^2 + 30m - 16$

iv) As $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$

$\frac{f}{g}(0) = \frac{f(0)}{g(0)}$

$= \frac{2}{-8}$

$= -\frac{1}{4}$

2.1.3 Composite function:

Let f:A→B and g: B→C where $f(x) = y$ and $g(y)$ = z, x∈A, y∈B, z∈C. We define a function h:A→C such that $h(x) = z$ then the function h is called composite function of f and g and is denoted by gof.

$\therefore gof$: A→C

$gof(x) = g[f(x)]$

It is also called function of a function.

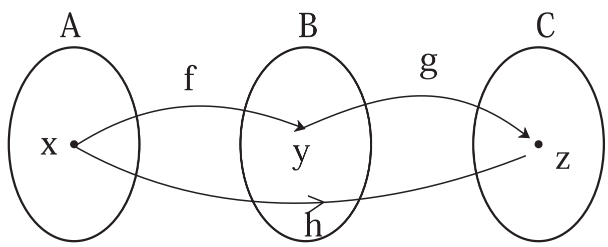


Fig. 2.21

e.g.

Let A = {1,2,3,4} and B = {1,3,5,7,10}

C = {–2,6,22,46,97,100} where

f = {(1,1), (2,3), (3,5), (4,7)} and

g = {(1,–2), (3,6), (5,22), (7,46), (10,97)}

$\therefore gof$ = {(1,-2), (2,6), (3,22), (4,46)}

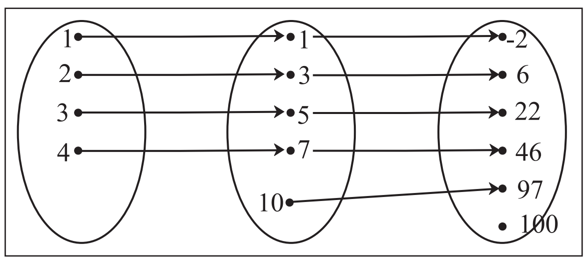


Fig. 2.22

Since

$f = \{(x,y) / x \in A, y \in B, \text{ and } y = 2x - 1\}$

$g = \{(y,z) / y \in B, z \in C, \text{ and } z = y^2 - 3\}$

then

$gof(x) = \{(x,z)/ x \in A, z \in C\}$ and

$z = (2x - 1)^2 - 3$

Ex 1. If $f(x) = \dfrac{2}{x+5}$ and $g(x) = x^2 - 1$, then find

i) $(f \circ g)(x)$ ii) $(g \circ f)(3)$

Solution :

i) As $(f \circ g)(x) = f[g(x)]$ and $f(x) = \dfrac{2}{x+5}$

Replace x from $f(x)$ by $g(x)$, to get

$$(f \circ g)(x) = \frac{2}{g(x)+5} = \frac{2}{x^2-1+5} = \frac{2}{x^2+4}$$

ii) As $(g \circ f)(x) = g[f(x)]$ and $g(x) = x^2 - 1$

Replace x by $f(x)$ in $g(x)$, to get

$$(g \circ f)(x) = [f(x)]^2 - 1$$

$$= \left(\frac{2}{x+5}\right)^2 - 1$$

Now let $x = 3$

$$(g \circ f)(3) = \left(\frac{2}{3+5}\right)^2 - 1 = \left(\frac{2}{8}\right)^2 - 1 = \left(\frac{1}{4}\right)^2 - 1 = \frac{1-16}{16} = -\frac{15}{16}$$

Ex 2. If $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \dfrac{1}{x}$, find $(g \circ f \circ h)(x)$

→ As $(g \circ f \circ h)(x) = g\{f[h(x)]\}$ and

$g(x) = x + 5$

Replace x in $g(x)$ by $f[h(x)] + 5$ to get

$(g \circ f \circ h)(x) = f[h(x)] + 5$

Now $f(x) = x^2$, so replace x in $f(x)$ by $h(x)$, to get

$(g \circ f \circ h)(x) = f[h(x)]^2 + 5$

Now $h(x) = \dfrac{1}{x}$ therefore,

$$(g \circ f \circ h)(x) = \left(\frac{1}{x}\right)^2 + 5 = \frac{1}{x^2} + 5$$

2.1.4 : Inverse functions:

Let $f : A \to B$ be one-one and onto function and $f(x) = y$ for $x \in A$. The inverse function $f^{-1} : B \to A$ is defined as $f^{-1}(y) = x$ for $y \in B$.

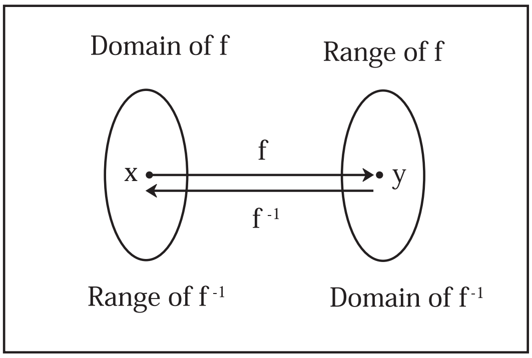


Fig. 2.23

Note: As f is one-one and onto every element $y \in B$ has a unique element $x \in A$ associated with y. Also note that $f \circ f^{-1}(x) = x$

Ex. 1 If f is one-one onto function with $f(3) = 7$ then $f^{-1}(7) = 3$.

Ex. 2. If f is one-one onto function with

$f(x) = 9 - 5x$, find $f^{-1}(-1)$.

→ Let $f^{-1}(-1) = m$, then $-1 = f(m)$

Therefore,

$$-1 = 9 - 5m$$

$$5m = 9 + 1$$

$$5m = 10$$

$$m = 2$$

That is $f(2) = -1$, so $f^{-1}(-1) = 2$.

Ex.3 Verify that $f(x) = \frac{x-5}{8}$ and $g(x) = 8x + 5$ are inverse functions.

As $f(x) = \frac{x-5}{8}$, replace x in $f(x)$ with $g(x)$

$$f[g(x)] = \frac{g(x)-5}{8} = \frac{8x+5-5}{8} = \frac{8x}{8} = x$$

and $g(x) = 8x + 5$, replace x in $g(x)$ with $f(x)$

$$g[f(x)] = 8f(x) + 5 = 8\left[\frac{x-5}{8}\right] + 5 = x - 5 + 5 = x$$

As $f[g(x)] = x$ and $g[f(x)] = x$, f and g are inverse functions.

Ex. 4: Determine whether the function

$f(x) = \frac{2x+1}{x-3}$ has inverse, if it exists find it.

f^{-1} exists only if is one-one and onto.

Solution : Consider $f(x_1) = f(x_2)$,

Therefore,

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3}$$

$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$

$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3$$

$$-6x_1 + x_2 = -6x_2 + x_1$$

$$7x_2 = 7x_1$$

$$x_2 = x_1$$

Hence, f is one-one function.

Let $f(x) = y$, so $y = \frac{2x+1}{x-3}$

Express x as function of y, as follows

$$y = \frac{2x+1}{x-3}$$

$$y(x-3) = 2x + 1$$

$$xy - 3y = 2x + 1$$

$$xy - 2x = 3y + 1$$

$$x(y-2) = 3y + 1$$

$$x = \frac{3y+1}{y-2}$$

Here the range of $f(x)$ is $R - \{2\}$.

x is defined for all y in the range.

Therefore $f(x)$ is onto function.

As function is one-one and onto, so f^{-1} exists.

As $f(x) = y$, so $f^{-1}(y) = x$

Therefore, $f^{-1}(y) = \frac{3y+1}{y-2}$

Replace x by y, to get

$f^{-1}(x) = \frac{3x+1}{x-2}$.

2.1.5 :Some Special Functions

i) Signum function : The function $f: R \to R$

$$f(x) = \begin{cases} -1 \text{ if } x<0 \\ 0 \text{ if } x=0 \\ 1 \text{ if } x>0 \end{cases}$$

is called the signum function. It is denoted as sgn (x)

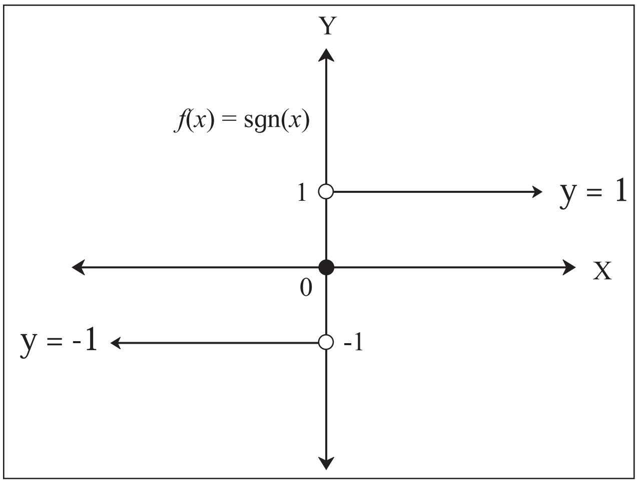


Fig. 2.24

Domain: R **Range:** $\{-1, 0, 1\}$

ii) Absolute value function (Modulus function):

Definition: Let f : R→R . $f(x) = |x|, \forall x \in R$

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

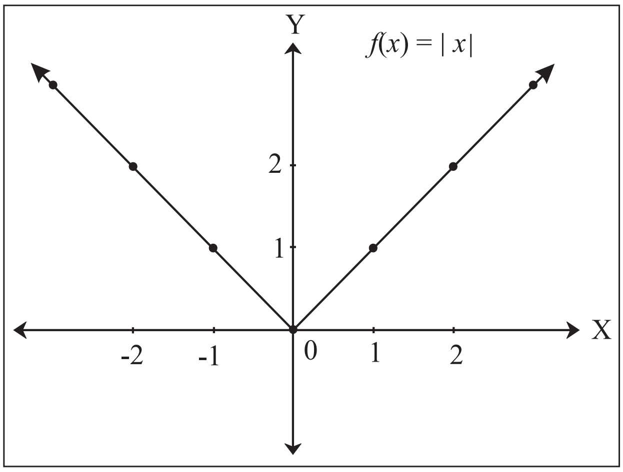


Fig. 2.25

Domain : R or $(-\infty,\infty)$ and **Range :** $[0,\infty)$

iii) Greatest Integer Function (Step Function):

Definition: For every real x , $f(x) = [x]$ = The greatest integer less than or equal to x.

$f(x) = n$, if $n \leq x < n + 1$, $x \in [n, n+1)$

We have

$$f(x) = \begin{cases} -2 & \text{if } -2 \leq x < -1 & \text{or } x \in [-2,-1) \\ -1 & \text{if } -1 \leq x < 0 & \text{or } x \in [-1,0) \\ 0 & \text{if } 0 \leq x < 1 & \text{or } x \in [0,1) \\ 1 & \text{if } 1 \leq x < 2 & \text{or } x \in [1,2) \\ 2 & \text{if } 2 \leq x < 3 & \text{or } x \in [2,3) \end{cases}$$

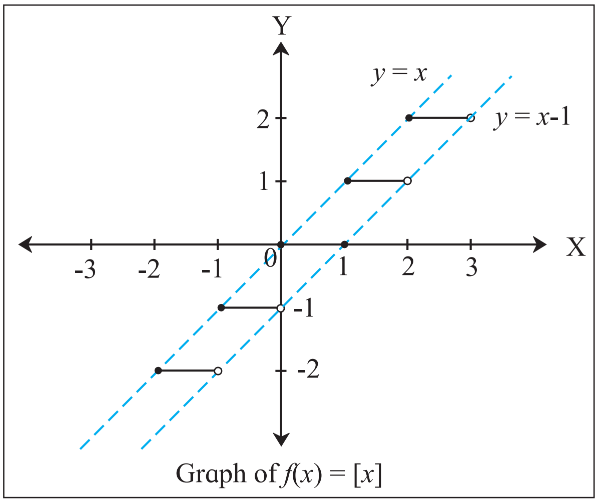


Fig. 2.26

Domain = R and **Range** = I (Set of integers)

EXERCISE 2.1

1) Check if the following relations are functions.

(a)

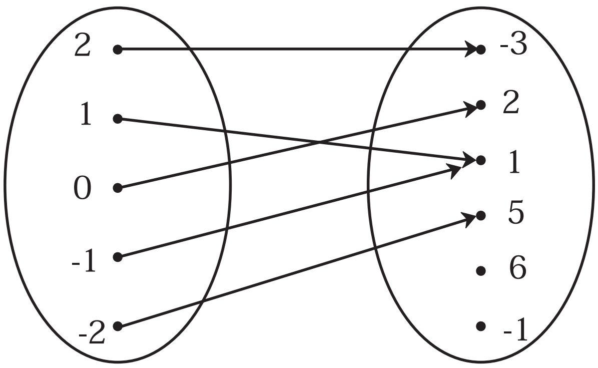


Fig. 2.27

(b)

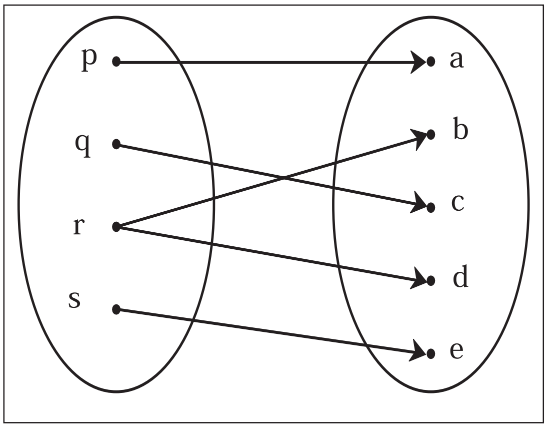


Fig. 2.28

(c)

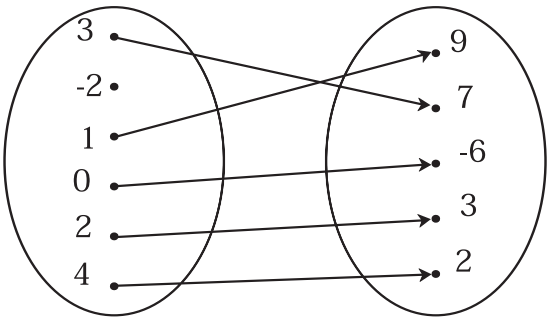


Fig. 2.29

2) Which sets of ordered pairs represent functions from A = {1,2,3,4} to B = {−1,0,1,2,3}? Justify.

(a) {(1,0), (3,3), (2,−1), (4,1), (2,2)}

(b) {(1,2), (2,−1), (3,1), (4,3)}

(c) {(1,3), (4,1), (2,2)}

(d) {(1,1), (2,1), (3,1), (4,1)}

3) If $f(m) = m^2 - 3m + 1$, find

(a) $f(0)$ (b) $f(-3)$

(c) $f\left(\frac{1}{2}\right)$ (d) $f(x+1)$

(e) $f(-x)$

4) Find x, if $g(x) = 0$ where

(a) $g(x) = \frac{5x-6}{7}$ (b) $g(x) = \frac{18-2x^2}{7}$

(c) $g(x) = 6x^2 + x - 2$

5) Find x, if $f(x) = g(x)$ where

$f(x) = x^4 + 2x^2$, $g(x) = 11x^2$

6) If $f(x) = \begin{cases} x^2+3, & x \le 2 \\ 5x+7, & x > 2 \end{cases}$, then find

(a) $f(3)$ (b) $f(2)$ (c) $f(0)$

7) If $f(x) = \begin{cases} 4x-2, & x \le -3 \\ 5, & -3 < x < 3 \\ x^2, & x \ge 3 \end{cases}$, then find

(a) $f(-4)$ (b) $f(-3)$

(c) $f(1)$ (d) $f(5)$

8) If $f(x) = 3x + 5$, $g(x) = 6x - 1$, then find

(a) $(f+g)(x)$ (b) $(f-g)(2)$

(c) $(fg)(3)$ (d) $(f/g)(x)$ and its domain.

9) If $f(x) = 2x^2 + 3$, $g(x) = 5x - 2$, then find

(a) $f \circ g$ (b) $g \circ f$

(c) $f \circ f$ (d) $g \circ g$

Let's remember!

- A Function from set A to the set B is a relation which associates every element of set A to unique element of set B and is denoted by $f : A \to B$.
- If f is a function from set A to the set B and if $(x, y) \in f$ then y is called the image of x under f and x is called the pre-image of y under f.

- A function $f : A \rightarrow B$ is said to be one-one if distinct elements in A have distinct images in B.
- A function $f : A \rightarrow B$ is said to be onto if every element of B is image of some element of A under the function f.
- A function $f : A \rightarrow B$ is such that there exists atleast one element in B which does not have pre-image in A then f is said to be an into function.
- If $f : A \rightarrow B$ and $g : B \rightarrow C$ with $f(x) = y$ and $g(y) = z$, $x \in A$, $y \in B$, $z \in C$ then define $h : A \rightarrow C$ such that $h(x) = z$, then the function h is called composite function of f and g and is denoted by $g\,of$.
- If $f : A \rightarrow B$ is one-one and onto, $g : B \rightarrow A$ is one-one and onto such that $g\,of : A \rightarrow A$ and $fog : B \rightarrow B$ are both identity functions then f and g are inverse functions of each other.
- Domain of f = Range of f^{-1} and

  Range of f = Domain of f^{-1}

MISCELLANEOUS EXERCISE -2

1) Which of the following relations are functions? If it is a function determine its domain and range.

   i) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

   ii) {(0, 0), (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3), (16, 4), (16, -4)}

   iii) {(1, 1), (3, 1), (5, 2)}

2) A function f: R→R defined by $f(x) = \frac{3x}{5} + 2$, $x \in R$. Show that f is one-one and onto. Hence find f^{-1}.

3) A function f is defined as follows

   $f(x) = 4x + 5$, for $-4 \leq x < 0$.

   Find the values of $f(-1)$, $f(-2)$, $f(0)$, if they exist.

4) A function f is defined as follows:

   $f(x) = 5 - x$ for $0 \leq x \leq 4$

   Find the value of x such that

   $f(x) = 3$

5) If $f(x) = 3x^2 - 5x + 7$ find $f(x - 1)$.

6) If $f(x) = 3x + a$ and $f(1) = 7$ find a and $f(4)$.

7) If $f(x) = ax^2 + bx + 2$ and $f(1) = 3$, $f(4) = 42$. find a and b.

8) If $f(x) = \frac{2x-1}{5x-2}$, $x \neq \frac{2}{5}$

   Verify whether $(fof)(x) = x$.

9) If $f(x) = \frac{x+3}{4x-5}$, $g(x) = \frac{3+5x}{4x-1}$

   then verify that $(fog)(x) = x$.

ACTIVITIES

Activity 2.1 :

If $f(x) = \frac{x+3}{x-2}$, $g(x) = \frac{2x+3}{x-1}$ Verify whether $fog(x) = gof(x)$.

Activity 2.2 :

$f(x) = 3x^2 - 4x+2$, $n \in \{0, 1, 2, 3, 4\}$ then represent the function as

i) By arrow diagram

ii) Set of ordered pairs

iii) In tabular form

iv) In graphical form

Activity 2.3 :

If $f(x) = 5x - 2$, $x > 0$, find $f^{-1}(x)$, $f^{-1}(7)$, for what value of x is $f(x) = 0$.

❖ ❖ ❖

3. COMPLEX NUMBERS

Let's study.

- Complex number
- Algebra of complex number
- Solution of Quadratic equation
- Cube roots of unity

Let's recall.

- Algebra of real numbers
- Solution of linear and quadratic equations
- Representation of a real number on the number line

Introduction:

Consider the equation $x^2 + 1 = 0$. This equation has no solution in the set of real numbers because there is no real number whose square is negative. To extend the set of real numbers to a larger set, which would include such solutions.

We introduce the symbol i such that $i = \sqrt{-1}$ and $i^2 = -1$.

Symbol i is called as an **imaginary unit.**

Swiss mathematician Euler (1707-1783) was the first mathematician to introduce the symbol i with $i^2 = -1$.

3.1 IMAGINARY NUMBER :

A number of the form ki, where $k \in$ R, $k \neq 0$ and $i = \sqrt{-1}$ is called an imaginary number.

For example

$\sqrt{-25} = 5i$, $2i$, $\frac{2}{7}\ i$, $-11i$, $\sqrt{-4}$ etc.

Let's Note.

The number i satisfies following properties,

i) $i \times 0 = 0$

ii) If $a \in$ R, then $\sqrt{-a^2} = \sqrt{i^2 a^2} = \pm\, ia$

iii) If a, $b \in$ R, and $ai = bi$ then $a = b$

3.2 COMPLEX NUMBER :

Definition : A number of the form $a+ib$, where a, $b \in$ R and $i = \sqrt{-1}$ is called a complex number and it is denoted by z.

$\therefore \quad z = a+ib = a + bi$

Here a is called the real part of z and is denoted by **Re(z) or R(z)**

'b' is called the imaginary part of z and is denoted by **Im(z) or I(z)**

The set of complex numbers is denoted by C

$\therefore$ C = $\{a+ib \mid a, b \in \text{R}, \text{ and } i = \sqrt{-1}\}$

For example

z	a+ib	Re(z)	Im(z)
$2+4i$	$2+4i$	2	4
$5i$	$0+5i$	0	5
$3-4i$	$3-4i$	3	-4
$5+\sqrt{-16}$	$5+4i$	5	4
$2+\sqrt{-5}$	$2+\sqrt{5}\ i$	2	$\sqrt{5}$
$7+\sqrt{3}$	$(7+\sqrt{3})+0i$	$(7+\sqrt{3})$	0

Let's Note.

1) A complex number whose real part is zero is called a imaginary number. Such a number is of the form z = 0 + *ib* = *ib* = *bi*

2) A complex number whose imaginary part is zero is a real number.

   $z = a + 0i = a$, for every real number.

3) A complex number whose both real and imaginary parts are zero is the zero complex number.

   $0 = 0 + 0i$

4) The set R of real numbers is a subset of the set C of complex numbers.

3.2.1 Conjugate of a Complex Number :

Definition : The conjugate of a complex number z=a+*i*b is defined as a − *i*b and is denoted by $\overline{z}$

For example

z	$\overline{z}$
$3 + 4i$	$3 - 4i$
$7i - 2$	$-7i - 2$
3	3
$5i$	$-5i$
$2 + \sqrt{3}$	$2 + \sqrt{3}$
$7 + \sqrt{-5}$	$7 - \sqrt{5}\, i$

Properties of $\overline{z}$

1) $(\overline{\overline{z}}) = z$

2) If $z = \overline{z}$, then z is real.

3) If $z = -\overline{z}$, then z is imaginary.

Now we define the four fundamental operations of addition, subtraction, multiplication and division of complex numbers.

3.3 ALGEBRA OF COMPLEX NUMBERS :

3.3.1 Equality of two Complex Numbers :

Definition : Two complex numbers $z_1 = a+ib$ and $z_2 = c + id$ are said to be equal if their real and imaginary parts are equal, i.e. $a = c$ and $b = d$.

For example,

i) If $x + iy = 4 + 3i$ then $x = 4$ and $y = 3$

1) Addition :

let $z_1 = a+ib$ and $z_2 = c+id$

then $z_1 + z_2 = a+ib + c+id$

$= (a+c) + (b+d)\, i$

Hence, $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Ex. 1) $(2 + 3i) + (4 + 3i) = (2+4) + (3+3)i$

$= 6 + 6i$

2) $(-2 + 5i) + (7 + 3i) + (6 - 4i)$

$= [(-2) + 7 + 6] + [5 + 3 + (-4)]i$

$= 11 + 4i$

Properties of addition : If z_1, z_2, z_3 are complex numbers then

i) $z_1 + z_2 = z_2 + z_1$

ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

iii) $z_1 + 0 = 0 + z_1 = z_1$

iv) $z + \overline{z} = 2\text{Re}(z)$

v) $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$

2) Scalar Multiplication :

If $z = a+ib$ is any complex number, then for every real number k, we define $kz = ka + i(kb)$

Ex. 1) If $z = 7 + 3i$ then

$5z = 5(7 + 3i) = 35 + 15i$

3) Subtraction :

Let $z_1 = a+ib$, $z_2 = c+id$ then

$z_1 - z_2 = z_1 + (-z_2) = (a+ib) + (-c-id)$

[Here $-z_2 = -1(z_2)$]

$= (a-c) + i\ (b-d)$

Hence,

$Re(z_1 - z_2) = Re(z_1) - Re(z_1)$

$Im(z_1 - z_2) = Im(z_1) - Im(z_2)$

Ex.1) $z_1 = 4+3i,\ z_2 = 2+i$

$\therefore z_1 - z_2 = (4+3i) - (2+i)$

$= (4-2) + (3-1)\ i$

$= 2 + 2i$

Ex. 2) $z_1 = 7+i,\ z_2 = 4i,\ z_3 = -3+2i$

then $2z_1 - (5z_2 + 2z_3)$

$= 2(7+i) - [5(4i) + 2(-3+2i)]$

$= 14 + 2i - [20i - 6 + 4i]$

$= 14 + 2i - [-6 + 24i]$

$= 14 + 2i + 6 - 24i$

$= 20 - 22i$

4) Multiplication :

Let $z_1 = a+ib$ and $z_2 = c+id$. We denote multiplication of z_1 and z_2 as $z_1.z_2$ and is given by

$z_1.z_2 = (a+ib)(c+id) = a(c+id)+ib(c+id)$

$= ac + adi + bci + i^2 bd$

$= ac + (ad+bc)i - bd\ (\because i^2 = -1)$

$z_1.z_2 = (ac-bd) + (ad+bc)i$

Ex. $z_1 = 2+3i,\ z_2 = 3-2i$

$\therefore z_1.z_2 = (2+3i)(3-2i) = 2(3-2i) + 3i(3-2i)$

$= 6 - 4i + 9i - 6i^2$

$= 6 - 4i + 9i + 6 \qquad (\because i^2 = -1)$

$= 12 + 5i$

Properties of Multiplication : If z_1, z_2, z_3 are complex numbers, then

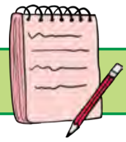
i) $z_1.z_2 = z_2.z_1$

ii) $(z_1.z_2).z_3 = z_1.(z_2.z_3)$

iii) $(z_1.1) = 1.z_1 = z_1$

iv) $z.\bar{z}$ is real number.

v) $\left(\overline{z_1.z_2}\right) = \bar{z}_1 . \bar{z}_2$ (Verify)



Let's Note.

If $z = a+ib$ then $z.\bar{z} = a^2 + b^2$

We have,

$i = \sqrt{-1}$, $\quad i^2 = -1$,

$i^3 = -i$, $\quad i^4 = 1$

Powers of *i* :

In general,

$i^{4n} = 1$, $\quad i^{4n+1} = i$,

$i^{4n+2} = -1$, $\quad i^{4n+3} = -i$ where $n \in N$

5) Division :

Let $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers such that $z_2 \neq 0$

Now,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \quad \text{where } z_2 \neq 0 \text{ i.e. } c+id \neq 0$$

The division can be carried out by multiplying and dividing $\frac{z_1}{z_2}$ by conjugate of $c+id$.

SOLVED EXAMPLES

Ex. 1) If $z_1 = 3+2i$, and $z_2 = 1+i$,

then write $\frac{z_1}{z_2}$ = in the form $a + ib$

Solution : $\frac{3+2i}{1+i} = \frac{3+2i}{1+i} \times \frac{1-i}{1-i}$

$$= \frac{3-3i+2i-2i^2}{(1)^2-(i)^2}$$

$$= \frac{3-3i+2i+2}{1+1} \qquad \because (i^2=-1)$$

$$= \frac{5-i}{2}$$

$$= \frac{5}{2} - \frac{1}{2}i$$

Ex. 2) Express $(1+2i)(-2+i)$ in the form of $a+ib$ where $a, b \in$ R.

Solution : $(1+2i)(-2+i) = -2+i-4i+2i^2$

$= -2-3i-2$

$= -4-3i$

Ex. 3) Write $(1+2i)(1+3i)(2+i)^{-1}$ in the form $a+ib$

Solution :

$$(1+2i)(1+3i)(2+i)^{-1} = \frac{(1+2i)(1+3i)}{2+i}$$

$$= \frac{1+3i+2i+6i^2}{2+i}$$

$$= \frac{-5+5i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-10+5i+10i-5i^2}{4-i^2}$$

$$= \frac{-5+15i}{4+1} \qquad \because (i^2=-1)$$

$$= \frac{-5+15i}{5}$$

$= -1 + 3i$

Ex. 4) Express $\frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$ in the form of $a+ib$

Solution : We know that, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$\therefore \frac{1}{i} + \frac{2}{i^2} + \frac{3}{i^3} + \frac{5}{i^4}$$

$$= \frac{1}{i} + \frac{2}{-1} + \frac{3}{-i} + \frac{5}{1}$$

$$= \frac{1}{i} - \frac{3}{i} - 2 + 5$$

$$= \frac{-2}{i} + 3 = 2i + 3 \text{ (verify!)}$$

Ex. 5) If a and b are real and $(i^4+3i)a + (i-1)b + 5i^3 = 0$, find a and b.

Solution : $(i^4+3i)a + (i-1)b + 5i^3 = 0+0i$

i.e. $(1+3i)a + (i-1)b - 5i = 0+0i$

$\therefore$ $a + 3ai + bi - b - 5i = 0+0i$

i.e. $(a-b) + (3a+b-5)i = 0+0i$

Equating real and imaginary parts, we get

$a-b = 0$ and $3a+b-5 = 0$

$\therefore$ $a=b$ and $3a+b = 5$

$\therefore$ $3a+a = 5$

i.e. $4a = 5$

or $a = \frac{5}{4}$

$\therefore$ $a = b = \frac{5}{4}$

Ex. 6) If $x + 2i + 15i^6y = 7x + i^3(y+4)$ find $x + y$, given that $x, y \in$ R.

Solution :

$x + 2i + 15i^6y = 7x + i^3(y+4)$

$\therefore$ $x + 2i - 15y = 7x - (y+4)i$

$\therefore$ $x - 15y + 2i = 7x - (y+4)i$

Equating real and imaginary parts, we get

$x - 15y = 7x$ and $2 = -(y+4)$

$\therefore$ $-6x - 15y = 0$ .... (i) $y+6 = 0$ ....(ii)

$\therefore$ $y = -6$, $x = 15$

$\therefore$ $x + y = 15 - 6 = 9$

Ex. 7) Find the value of $x^3 - x^2 + 2x + 4$ when $x = 1 + \sqrt{3}\ i$.

Solution : Since $x = 1 + \sqrt{3}\ i$

$\therefore (x-1) = \sqrt{3}\ i$

squaring both sides, we get

$(x-1)^2 = (\sqrt{3}\ i)^2$

$\therefore x^2 - 2x + 1 = 3i^2$

i.e. $x^2 - 2x + 1 = -3$

$\therefore x^2 - 2x + 4 = 0$

Now, consider

$x^3 - x^2 + 2x + 4 = x(x^2 - x + 2) + 4$

$= x(x^2 - 2x + 4 + x - 2) + 4$

$= x(0 + x - 2) + 4$

$= x^2 - 2x + 4$

$= 0$

Ex. 8) Show that $\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^3 = i.$

Solution :

$$\text{L.H.S.} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^3 = \left(\frac{\sqrt{3}+i}{2}\right)^3$$

$$= \frac{\left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2 i + 3\left(\sqrt{3}\right)i^2 + (i)^3}{(2)^3}$$

$$= \frac{3\sqrt{3} + 9i - 3\sqrt{3} - i}{8}$$

$$= \frac{8i}{8}$$

$= i$

$=$ R.H.S.

Ex. 9) If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 64$.

Solution : $x = -5 + 2\sqrt{-4}$

$\therefore x = -5 + 4i$

$\therefore x + 5 = 4i$

On squaring both sides

$(x+5)^2 = (4i)^2$

$\therefore x^2 + 10x + 25 = -16$

$\therefore x^2 + 10x + 41 = 0$

$$\begin{array}{r|l} & x^2 - x + 4 \\ \hline x^2 + 10x + 41 & x^4 + 9x^3 + 35x^2 - x + 64 \\ & x^4 + 10x^3 + 41x^2 \\ \hline & -x^3 - 6x^2 - x + 64 \\ & -x^3 - 10x^2 - 41x \\ \hline & 4x^2 + 40x + 64 \\ & 4x^2 + 40x + 164 \\ \hline & -100 \end{array}$$

$\therefore x^4 + 9x^3 + 35x^2 - x + 64$

$= (x^2 + 10x + 41)(x^2 - x + 4) - 100$

$= 0 \times (x^2 - x + 4) - 100$

$= -100$

EXERCISE 3.1

1) Write the conjugates of the following complex numbers

i) $3+i$ ii) $3-i$ iii) $-\sqrt{5} - \sqrt{7}\ i$

iv) $-\sqrt{-5}$ v) $5i$ vi) $\sqrt{5} - i$

vii) $\sqrt{2} + \sqrt{3}\ i$

2) Express the following in the form of a+ib, a, b$\in$R $i = \sqrt{-1}$. State the value of a and b.

i) $(1+2i)(-2+i)$ ii) $\dfrac{i(4+3i)}{(1-i)}$

iii) $\dfrac{(2+i)}{(3-i)(1+2i)}$ iv) $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i}$

v) $\dfrac{2+\sqrt{-3}}{4+\sqrt{-3}}$ vi) $(2+3i)(2-3i)$

vii) $\dfrac{4i^8 - 3i^9 + 3}{3i^{11} - 4i^{10} - 2}$

3) Show that $(-1+\sqrt{3}\, i)^3$ is a real number.

4) Evaluate the following :

i) i^{35} ii) i^{888} iii) i^{93} iv) i^{116}

v) i^{403} vi) $\dfrac{1}{i^{58}}$ vii) $i^{30} + i^{40} + i^{50} + i^{60}$

5) Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

6) Find the value of

i) $i^{49} + i^{68} + i^{89} + i^{110}$

ii) $i + i^2 + i^3 + i^4$

7) Find the value of $1 + i^2 + i^4 + i^6 + i^8 + ... + i^{20}$

8) Find the value of x and y which satisfy the following equations $(x, y \in R)$

i) $(x+2y) + (2x-3y)\, i + 4i = 5$

ii) $\dfrac{x+1}{1+i} + \dfrac{y-1}{1-i} = i$

9) Find the value of

i) $x^3 - x^2 + x + 46$, if $x = 2+3i$.

ii) $2x^3 - 11x^2 + 44x + 27$, if $x = \dfrac{25}{3-4i}$.

3.4 Square root of a complex number :

Consider $z = x+iy$ be any complex number

Let $\sqrt{x+iy} = a+ib,\ a, b \in R$

On squaring both the sides, we get

$x+iy = (a+ib)^2$

$x+iy = (a^2-b^2) + (2ab)\, i$

Equating real and imaginary parts, we get

$x = (a^2-b^2)$ and $y = 2ab$

Solving the equations simultaneously, we can get the values of a and b.

SOLVED EXAMPLES

1) Find the square root of $6+8i$

Let the square root of $6+8i$ be $a+ib$, $(a, b \in R)$

$\therefore \sqrt{6+8i} = a+ib,\ a, b \in R$

On squaring both the sides, we get

$6+8i = (a+ib)^2$

$\therefore 6+8i = a^2-b^2 + 2abi$

Equating real and imaginary parts, we have

$6 = a^2-b^2$ .... (1)

$8 = 2ab$ .... (2)

$\therefore a = \dfrac{4}{b}$

$\therefore 6 = \left(\dfrac{4}{b}\right)^2 - b^2$

i.e. $6 = \dfrac{16}{b^2} - b^2$

$\therefore b^4 + 6b^2 - 16 = 0$

put $b^2 = m$

$\therefore m^2 + 6m - 16 = 0$

$\therefore (m+8)(m-2) = 0$

$\therefore m = -8$ or $m = 2$

i.e. $b^2 = -8$ or $b^2 = 2$

but $b \in R$ $\therefore b^2 \neq -8$

$\therefore b^2 = 2$ $\therefore b = \pm\sqrt{2}$

when $b = \sqrt{2}$, $a = 2\sqrt{2}$

$\therefore$ Square root of

$6+8i = 2\sqrt{2} + \sqrt{2}\, i = \sqrt{2}\,(2+i)$

when $b = -\sqrt{2}$, $a = -2\sqrt{2}$

$\therefore$ Square root of

$6+8i = -2\sqrt{2} - \sqrt{2}\, i = -\sqrt{2}\,(2+i)$

$\therefore \sqrt{6+8i} = \pm\sqrt{2}\,(2+i)$

Ex. 2 : Find the square root of $2i$

Solution :

Let $\sqrt{2i} = a+ib \qquad a, b \in R$

On squaring both the sides, we have

$2i = (a+ib)^2$

$\therefore 0+2i = a^2-b^2+2iab$

Equating real and imaginary parts, we have

$a^2-b^2 = 0, \quad 2ab = 2, \quad ab = 1$

As $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$(a^2+b^2)^2 = 0^2 + 2^2$

$(a^2+b^2)^2 = 2^2$

$\therefore a^2+b^2 = 2$

Solving $a^2+b^2 = 2$ and $a^2-b^2 = 0$ we get

$2a^2 = 2$

$a^2 = 1$

$a = \pm 1$

But $b = \dfrac{2}{2a} = \dfrac{1}{a} = \dfrac{1}{\pm 1} = \pm 1$

$\therefore \sqrt{2i} = 1+i$ or $-1-i$

i.e. $\sqrt{2i} = \pm(1+i)$

3.5 Solution of a Quadratic Equation in complex number system :

Let the given equation be $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$

$\therefore$ the solution of this quadratic equation is given by

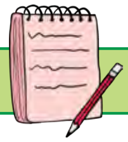
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Hence, the roots of the equation $ax^2+bx+c = 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

The expression $(b^2-4ac) = D$ is called the discriminant.

If $D < 0$ then the roots of the given quadratic equation are not real in nature, that is the roots of such equation are complex numbers.



Let's Note.

If $p + iq$ is a root of equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ then $p - iq$ is also a root of the given equation. That is complex roots occurs in conjugate pairs.

SOLVED EXAMPLES

Ex. 1 : Solve $x^2 + x + 1 = 0$

Solution : Given equation is $x^2 + x + 1 = 0$

where $a = 1, b = 1, c = 1$

So, the given equation has complex roots.

These roots are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore$ Roots are $\dfrac{-1+\sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$

Ex. 2 : Solve the following quadratic equation $x^2 - 4x + 13 = 0$

Solution : The given quadratic equation is

$x^2 - 4x + 13 = 0$

$x^2 - 4x + 4 + 9 = 0$

$\therefore (x-2)^2 + 3^2 = 0$

$(x-2)^2 = -3^2$

$(x-2)^2 = +3^2.i^2$

taking square root we get,

$(x-2) = \pm 3i$

$\therefore x = 2 \pm 3i$

$\therefore x = 2 + 3i$ or $x = 2 - 3i$

Solution set $= \{2 + 3i, 2 - 3i\}$

Ex. 3 : Solve $x^2 + 4ix - 5 = 0$; where $i = \sqrt{-1}$

Solution : Given quadratic equation is

$$x^2 + 4ix - 5 = 0$$

Compairing with $ax^2 + bx + c = 0$

$$a = 1, \quad b = 4i, \quad c = -5$$

Consider $b^2 - 4ac = (4i)^2 - 4(1)(-5)$

$= 16i^2 + 20$

$= -16 + 20 \quad (\because i^2 = -1)$

$= 4$

The roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4i \pm \sqrt{4}}{2(1)}$$

$$= \frac{-4i \pm 2}{2}$$

$$x = -2i \pm 1$$

Solution set = $\{-2i + 1, -2i - 1\}$

EXERCISE 3.2

1) Find the square root of the following complex numbers

i) $-8-6i$ ii) $7+24i$ iii) $1+4\sqrt{3}\,i$

iv) $3+2\sqrt{10}\,i$ v) $2(1-\sqrt{3}\,i)$

2) Solve the following quadratic equations.

i) $8x^2 + 2x + 1 = 0$

ii) $2x^2 - \sqrt{3}\,x + 1 = 0$

iii) $3x^2 - 7x + 5 = 0$

iv) $x^2 - 4x + 13 = 0$

3) Solve the following quadratic equations.

i) $x^2 + 3ix + 10 = 0$

ii) $2x^2 + 3ix + 2 = 0$

iii) $x^2 + 4ix - 4 = 0$

iv) $ix^2 - 4x - 4i = 0$

4) Solve the following quadratic equations.

i) $x^2 - (2+i)\,x - (1-7i) = 0$

ii) $x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}\,i = 0$

iii) $x^2 - (5-i)\,x + (18+i) = 0$

iv) $(2 + i)x^2 - (5-i)\,x + 2\,(1-i) = 0$

3.6 Cube roots of unity :

Number 1 is often called unity. Let x be a cube root of unity.

$\therefore\ x^3 = 1$

$\therefore\ x^3 - 1 = 0$

$\therefore\ (x-1)(x^2 + x + 1) = 0$

$\therefore\ x - 1 = 0$ or $x^2 + x + 1 = 0$

$\therefore\ x = 1$ or $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4\times1\times1}}{2\times1}$

$\therefore\ x = 1$ or $x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\therefore\ x = 1$ or $x = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ Cube roots of unity are, 1, $\dfrac{-1+i\sqrt{3}}{2}$, $\dfrac{-1-i\sqrt{3}}{2}$

Among the three cube roots of unity, one is real and other two roots are complex conjugates of each other.

Now consider

$$\left(\frac{-1+\sqrt{3}\,i}{2}\right)^2$$

$$= \frac{1}{4}\left[(-1)^2 + 2\times(-1)\,i\sqrt{3} + \left(i\sqrt{3}\right)^2\right]$$

$$= \frac{1}{4}\,(1 - 2i\sqrt{3} - 3)$$

$$= \frac{1}{4}\,(-2 - 2i\sqrt{3})$$

$$= \frac{-1-i\sqrt{3}}{2}$$

Similarly it can be verified that $\left(\dfrac{-1-i\sqrt{3}}{2}\right)^2$

$\dfrac{-1+i\sqrt{3}}{2}$

Thus complex roots of unit are squares of each other. Thus cube roots of unity are given by

$$1, \frac{-1+i\sqrt{3}}{2}, \left(\frac{-1-i\sqrt{3}}{2}\right)^2$$

Let $\dfrac{-1+i\sqrt{3}}{2} = w$, then $\dfrac{-1-i\sqrt{3}}{2} = w^2$

Hence, cube roots of unity are 1, w, w^2 or 1, w, $\overline{w}$

where $w = \dfrac{-1+i\sqrt{3}}{2}$ and $w^2 = \dfrac{-1-i\sqrt{3}}{2}$

w is complex cube root of 1.

$\therefore\ w^3 = 1\ \ \therefore\ w^3 - 1 = 0$

i.e. $(w-1)(w^2+w+1) = 0$

$\therefore\ w=1$ or $w^2+w+1 = 0$

but $w \neq 1$

$\therefore\ w^2+w+1 = 0$

Properties of 1, w, w^2

i) $w^2 = \dfrac{1}{w}$ and $\dfrac{1}{w^2} = w$

ii) $w^3 = 1$ so $w^{3n} = 1$

iii) $w^4 = w^3 .w = w$ so $w^{3n+1} = w$

iv) $w^5 = w^2 .w^3 = w^2.1 = w^2$

So $w^{3n+2} = w^2$

v) $\overline{w} = w^2$

vi) $(\overline{w})^2 = w$

SOLVED EXAMPLES

Ex. 1 : If w is a complex cube root of unity, then prove that

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = -1$

ii) $(1+w^2)^3 = -1$

iii) $(1-w+w^2)^3 = -8$

Solution : Given, w is a complex cube root of unity.

$\therefore\ w^3 = 1$ Also $w^2+w+1 = 0$

$\therefore\ w^2+1 = -w$ and $w+1 = -w^2$

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = \dfrac{w+1}{w^2} = \dfrac{-w^2}{w^2} = -1$

ii) $(1+w^2)^3 = (-w)^3 = -w^3 = -1$

iii) $(1-w+w^2)^3 = (1+w^2-w)^3$

$= (-w-w)^3 \quad (\because\ 1+w^2 = -w)$

$= (-2w)^3$

$= -8w^3$

$= -8 \times 1$

$= -8$

Ex. 2 : If w is a complex cube root of unity, then show that

$(1-w)(1-w^2)(1-w^4)(1-w^5) = 9$

Solution : $(1-w)(1-w^2)(1-w^4)(1-w^5)$

$= (1-w)(1-w^2)(1-w^3.w)(1-w^3.w^2)$

$= (1-w)(1-w^2)(1-w)(1-w^2)$

$= (1-w)^2(1-w^2)^2$

$= [(1-w)(1-w^2)]^2$

$= (1-w^2-w+w^3)^2$

$= [1-(w^2+w)+1]^2$

$= [1-(-1)+1]^2$

$= (1+1+1)^2$

$= (3)^2$

$= 9$

Ex. 3 : Prove that

$1+w^n+w^{2n}=3$, if n is a multiple of 3

$1+w^n+w^{2n}=0$, if n is not multiple of 3, $n \in N$

Solution : If n is a multiple of 3 then n=3k and if n is not a multiple of 3 then n = 3k+1 or n = 3k+2, where $k \in N$

Case 1: If n is multiple of 3

then $1+w^n+w^{2n} = 1+w^{3k}+w^{2\times 3k}$

$= 1+(w^3)^k+(w^3)^{2k}$

$= 1+(1)^k+(1)^{2k}$

$=1+1+1$

$= 3$

Case 2: If n = 3k + 1

then $1+w^n+w^{2n} = 1+(w)^{3k+1}+(w^2)^{3k+1}$

$= 1+(w^3)^k.w +(w^3)^{2k}.w^2$

$= 1+(1)^k.w +(1)^{2k}.w^2$

$=1+w+w^2$

$= 0$

Similarly by putting $n = 3k+2$, we have,

$1+w^n+w^{2n}=0$. Hence the results.

EXERCISE 3.3

1) If w is a complex cube root of unity, show that

i) $(2-w)(2-w^2) =7$

ii) $(2+w+w^2)^3-(1-3w+w^2)^3 = 65$

iii) $\dfrac{(a+bw+cw^2)}{c+aw+bw^2} = w^2$

2) If w is a complex cube root of unity, find the value of

i) $w + \dfrac{1}{w}$ ii) $w^2+w^3+w^4$ iii) $(1+w^2)^3$

iv) $(1-w-w^2)^3 + (1-w+w^2)^3$

v) $(1+w)(1+w^2)(1+w^4)(1+w^8)$

3) If $\propto$ and β are the complex cube roots of unity, show that

$\propto^2+\beta^2+\propto\beta = 0$

4) If $x=a+b$, $y=\propto a+\beta b$, and $z=a\beta+b\propto$ where $\propto$ and β are the complex cube-roots of unity, show that $xyz =a^3+b^3$

5) If w is a complex cube-root of unity, then prove the following

i) $(w^2+w-1)^3 = -8$

ii) $(a+b)+(aw+bw^2)+(aw^2+bw)= 0$.

Let's remember!

* A number of the form a+ib, where a and b are real numbers, $i = \sqrt{-1}$, is called a complex number.

* Let $z_1 = a+ib$ and $z_2 = c+id$. Then

$z_1 + z_2 = (a+c) + (b+d)i$

$z_1 z_2 = (ac-bd) + (ad+bc)i$

* For any positive integer k,

$i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

* The conjugate of complex number $z = a+ib$ denoted by $\overline{z}$, is given by $\overline{z} = a-ib$

* The cube roots of unity are denoted by 1, w, w^2 or 1, w, $\overline{w}$

MISCELLANEOUS EXERCISE - 3

1) Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$

2) Find the value of $\sqrt{-3} \times \sqrt{-6}$

3) Simplify the following and express in the form a+ib.

i) $3+\sqrt{-64}$ ii) $(2i^3)^2$ iii) $(2+3i)(1-4i)$

iv) $\dfrac{5}{2}i(-4-3i)$ v) $(1+3i)^2(3+i)$ vi) $\dfrac{4+3i}{1-i}$

vii) $\left(1+\dfrac{2}{i}\right)\left(3+\dfrac{4}{i}\right)(5+i)^{-1}$ viii) $\dfrac{\sqrt{5}+\sqrt{3}i}{\sqrt{5}-\sqrt{3}i}$

ix) $\dfrac{3i^5+2i^7+i^9}{i^6+2i^8+3i^{18}}$ x) $\dfrac{5+7i}{4+3i}+\dfrac{5+7i}{4-3i}$

4) Solve the following equations for $x, y \in R$

i) $(4-5i)x+(2+3i)y=10-7i$

ii) $(1-3i)x+(2+5i)y=7+i$

iii) $\dfrac{x+iy}{2+3i}=7-i$

iv) $(x+iy)\ (5+6i)=2+3i$

v) $2x+i^9y\ (2+i)=xi^7+10i^{16}$

5) Find the value of

i) $x^3+2x^2-3x+21$, if $x=1+2i$.

ii) x^3-5x^2+4x+8, if $x=\dfrac{10}{3-i}$

iii) $x^3-3x^2+19x-20$, if $x=1-4i$.

6) Find the square roots of

i) $-16+30i$ ii) $15-8i$ iii) $2+2\sqrt{3}\,i$

iv) $18i$ v) $3-4i$ vi) $6+8i$

ACTIVITY

Activity 3.1:

Carry out the following activity.

If $x+2i=7x-15i^6y+i^3(y+4)$, find $x+y$.

Given : $x+2i=7x-15i^6y+i^3(y+4)$

$x+2i=7x-15\square y+\square(y+4)$

$x+2i=7x+\square y-(y+4)\square$

$x-\square+2i=7x-(y+4)\square$

$\therefore\ x-15y-7x=\square$ and $\square=-(y+4)$

$\therefore\ -6x-15y=\square$ and $y=\square$

$\therefore\ x=\square,\ y=\square$

$\therefore\ x+y=\square$

Activity 3.2:

Carry out the following activity

Find the value of $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$ in terms of w^2

Consider $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=\dfrac{\square\left(\dfrac{a}{\omega^2}+\dfrac{b}{\omega}+c\right)}{c+a\omega+b\omega^2}$

$=\dfrac{\left(\dfrac{a\square}{\omega^3}+\dfrac{b\square}{\omega^3}+c\right)\omega^2}{c+a\omega+b\omega^2}$

$=\dfrac{\left(\dfrac{a\omega}{1}+\dfrac{b\square}{\square}+c\right)\omega^2}{c+a\omega+b\omega^2}$

$=\dfrac{(\square+\square+c)\omega^2}{c+a\omega+b\omega^2}$

$=\square$

❖ ❖ ❖

4. SEQUENCES AND SERIES

Let's study.

- Geometric progression (G.P.)
- n^{th} term of a G.P.
- Sum of n terms of a G.P.
- Sum of infinite terms of a G.P.
- Sigma notation.

4.1 SEQUENCE :

A set of numbers, where the numbers are arranged in a definite order, is called a sequence.

Natural numbers is an example of a sequence.

In general,a sequence is written as $\{t_n\}$.

Finite sequence – A sequence containing finite number of terms is called a finite sequence.

Infinite sequence – A sequence is said to be infinite if it is not a finite sequence.

In this case for every positive integer n , there is a unique t_n in the sequence.

Sequences that follow specific rule are called progressions.

In the previous class, we have studied Arithmetic Progression (A.P.).

In a sequence if the difference between any term and its preceding term $(t_{n+1} - t_n)$ is constant, for all n∈N then the sequence is called an Arithmetic Progression (A.P.)

Consider the following sequences.

1) 2,5,8,11,14,…………..

2) 4,10,16,22,28,…………..

3) 4,16,64,256,…………….

4) $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}$,…………..

The sequences 1) and 2), are in A.P. because $t_{n+1} - t_n$ is constant.

But in sequences 3) and 4), the difference is not constant. In these sequences, the ratio of any term to its preceding term that is $\frac{t_{n+1}}{t_n}$ is constant [n∈N].

Such a sequence is called a 'GEOMETRIC PROGRESSION' (G. P.).

Let's learn.

4.2 GEOMETRIC PROGRESSION (G.P.)

Definition : A Sequence $\{t_n\}$ is said to be a Geometric Progression if $\frac{t_{n+1}}{t_n}$ = constant.

$\frac{t_{n+1}}{t_n}$ is called the common ratio of the G.P. and

it is denoted by r ($r \neq 0$) , for all $n \in$ N .

It is a convention to denote the first term of the geometric progression by a ($a \neq 0$) .

The terms of a geometric progression with first term 'a' and common ratio 'r' are as follows.

a, ar, ar^2, ar^3, ar^4,……………………

Let's see some examples of G.P.

(i) 2 , 8 , 32 , 128 , 512 , ………………… is a G.P. with a = 2 and r = 4.

(ii) 25, 5 , 1 , $\frac{1}{5}$, ………….................is a G.P. with a = 25 and r = $\frac{1}{5}$.

4.3 General term or the n^{TH} term of a G.P.

If a and r are the first term and common ratio of G.P. respectively, then its general term is given by $t_n = ar^{n-1}$.

Let's find n^{th} term of the following G.P.

i) 2,8,32,128,512,…………………

Here a=2 , r = 4

$t_n = ar^{n-1} = 2\,(4)^{n-1}$

ii) 25, 5 , 1 ,$\frac{1}{5}$,………….. ………………

Here a = 25 , r = $\frac{1}{5}$

$t_n = ar^{n-1} = 25\left(\frac{1}{5}\right)^{n-1}$

4.3.1 Properties of Geometric Progression.

i) Reciprocals of terms of a G.P. are also in G.P.

ii) If each term of a G.P. is multiplied or divided by a non zero constant ,then the resulting sequence is also a G.P.

iii) If each term of a G.P. is raised to the same power , the resulting sequence is also a G.P.

SOLVED EXAMPLES

Ex.1) For the following G.P.s find the n^{th} term

3,-6,12,-24,……………….

Solution:

Here a=3 , r = –2

$\therefore\ t_n = ar^{n-1} = 3\,(-2)^{n-1}$

Ex 2) Verify whether $1, \frac{-3}{2}, \frac{9}{4}, -\frac{-27}{8}, \ldots\ldots\ldots\ldots$ is a G.P. If it is a G.P., find its ninth term.

Solution : Here $t_1 = 1$, $t_2 = \frac{-3}{2}$, $t_3 = \frac{9}{4}$,

Consider $= \frac{t_2}{t_1} = \frac{\frac{-3}{2}}{1} = \frac{-3}{2}$

$\frac{t_3}{t_2} = \frac{\frac{9}{4}}{\frac{-3}{2}} = \frac{-3}{2}$

$\frac{t_4}{t_3} = \frac{\frac{-27}{8}}{\frac{9}{4}} = \frac{-3}{2}$

Here the ratio of any term to its previous term is constant hence the given sequence is a G.P.

Now $t_9 = ar^{9-1} = a\,r^{8} = 1\left(\frac{-3}{2}\right)^{8} = \frac{6561}{256}$.

Ex 3) For a G.P. if $a = 3$ and $t_7 = 192$, find r and t_{11} .

Solution : Given $a = 3$, $t_7 = ar^6 = 192$

$\therefore\ 3\,(r)^6 = 192$, $r^6 = \frac{192}{3} = 64$

$\therefore\ r^6 = 2^{6}$,

$\therefore\ r = \pm 2.$

$t_{11} = a\,r^{10} = 3\,(\pm 2)^{10} = 3(1024) = 3072.$

Ex 4) For a G.P. $t_3 = 486$, $t_6 = 18$, find t_{10}

Solution : We know that $t_n = ar^{n-1}$

$t_3 = ar^2 = 486$ ----------------- (1)

$t_6 = ar^5 = 18$ ----------------- (2)

dividing equation (2) by (1) we get,

$\frac{t_6}{t_3} = \frac{ar^5}{ar^2} = \frac{18}{486} = \frac{2}{54} = \frac{1}{27}$

$r^3 = \left(\frac{1}{3}\right)^3$, $r = \frac{1}{3}$.

Now from (1) $ar^2 = 486$

$$a\left(\frac{1}{3}\right)^2 = 486.$$

$$a\left(\frac{1}{9}\right) = 486,$$

$$a = 486 \times 9$$

$$a = 4374.$$

Now $t_{10} = ar^9 = 4374\left(\frac{1}{3}\right)^9$

$$= \frac{243 \times 2 \times 9}{3^9} = \frac{3^5 \times 2 \times 3^2}{3^5 \times 3^2 \times 3^2}$$

$$\therefore t_{10} = \frac{2}{9}$$

Ex 5) If for a sequence $\{t_n\}$, $t_n = \frac{5^{n-2}}{4^{n-3}}$, show that the sequence is a G.P.

Find its first term and the common ratio.

Solution: $t_n = \frac{5^{n-2}}{4^{n-3}}$

$$t_{n+1} = \frac{5^{n-1}}{4^{n-2}}$$

Consider $\frac{t_{n+1}}{t_n} = \frac{\frac{5^{n-1}}{4^{n-2}}}{\frac{5^{n-2}}{4^{n-3}}}$

$$= \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}} = \frac{5}{4} = \text{constant},$$

$\forall$ n $\in$ N.

The given sequence is a G.P. with $r = \frac{5}{4}$ and $t_1 = a = \frac{5^{1-2}}{4^{1-3}} = \frac{5^{-1}}{4^{-2}} = \frac{16}{5}$.

Let's Note.

i) To find 3 numbers in G.P., it is convenient to take the numbers as $\frac{a}{r}, a, ar$

ii) 4 numbers in a G.P. as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$, (here the ratio is r^2)

iii) 5 numbers in a G.P. as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Ex 6) Find three numbers in G.P. such that their sum is 42 and their product is 1728.

Solution: Let the three numbers be $\frac{a}{r}, a, ar$.

According to first condition their sum is 42

$$\therefore \frac{a}{r} + a + ar = 42$$

$$a\left(\frac{1}{r} + 1 + r\right) = 42$$

$$\therefore \frac{1}{r} + 1 + r = \frac{42}{a}$$

$$\frac{1}{r} + r = \frac{42}{a} - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the second condition their product is 1728

$$\frac{a}{r} \cdot a \cdot ar = 1728$$

$$\therefore a^3 = 1728 = (12)^3$$

$$\therefore a = 12 .$$

substitute a = 12 in equation (1), we get

$$\frac{1}{r} + r = \frac{42}{12} - 1$$

$$\frac{1}{r} + r = \frac{42-12}{12}$$

$$\frac{1}{r} + r = \frac{30}{12}$$

$$\frac{1+r^2}{r} = \frac{5}{2}$$

$$\therefore 2 + 2r^2 = 5r$$

$$\therefore 2r^2 - 5r + 2 = 0$$

$$\therefore (2r-1)(r-2) = 0$$

$$\therefore 2r = 1 \text{ or } r = 2$$

$$\therefore r = \frac{1}{2} \text{ or } r = 2$$

Now if a=12, and $r = \frac{1}{2}$ then the required numbers are 24 , 12 , 6.

If $a = 12$, and $r = 2$ then the required numbers are 6,12,24.

$\therefore$ 24,12,6 or 6,12,24 are the three required numbers in G.P.

Ex 7) In a G.P. ,if the third term is $\frac{1}{5}$ and sixth term is $\frac{1}{625}$, find its n^{th} term .

Solution : Here $t_3 = \frac{1}{5}$, $t_6 = \frac{1}{625}$

$$t_3 = ar^2 = \frac{1}{5} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$t_6 = ar^5 = \frac{1}{625} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Divide equation (2) by equation (1)

we get,

$$r^3 = \frac{1}{125} = \frac{1}{5^3}$$

$\therefore r = \frac{1}{5}$. Substitute $r = \frac{1}{5}$ in equation (1)

we get

$$a\left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

$\therefore$ a = 5.

$$t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = 5 \times 5^{1-n} = 5^{2-n}.$$

Ex 8) Find four numbers in G. P. such that their product is 64 and sum of the second and third number is 6.

Solution : Let the four numbers be $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 (common ratio is r^2)

According to the first condition

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$\therefore a^4 = 64$

$\therefore a = 2\sqrt{2}$.

Now using second condition $\frac{a}{r} + ar = 6$

$\frac{2\sqrt{2}}{r} + 2\sqrt{2}r = 6$. dividing by 2 we get ,

$\frac{\sqrt{2}}{r} + \sqrt{2}\,r = 3$ now multiplying by r we get

$$\sqrt{2} + \sqrt{2}\,r^2 - 3r = 0$$

$$\sqrt{2}\,r^2 - 3r + \sqrt{2} = 0,$$

$$\sqrt{2}\,r^2 - 2r - r + \sqrt{2} = 0,$$

$$\sqrt{2}\,r(r - \sqrt{2}) - 1(r - \sqrt{2}) = 0.$$

$$r = \sqrt{2} \text{ or } r = \frac{1}{\sqrt{2}} .$$

If $a = 2\sqrt{2}$, and $r = \sqrt{2}$ then 1 ,2 , 4, 8 are the four required numbers

If $a = 2\sqrt{2}$, and $r = \frac{1}{\sqrt{2}}$ then 8 ,4, 2, 1 are the four required numbers in G.P.

Ex 9) If p,q,r,s are in G.P. then show that

$$(q-r)^2 + (r-p)^2 + (s-q)^2 = (p-s)^2$$

Solution : As p,q,r,s are in G.P. $\frac{q}{p} = \frac{r}{q} = \frac{s}{r} = k$ (say)

$\therefore q^2 = pr$, $r^2 = qs$, $qr = ps$

consider L.H.S.

$= (q-r)^2 + (r-p)^2 + (s-q)^2$

$= q^2 - 2qr + r^2 + r^2 - 2rp + p^2 + s^2 - 2sq + q^2$

$= pr - 2qr + qs + qs - 2rp + p^2 + s^2 - 2sq + pr$

$= -2qr + p^2 + s^2 = -2ps + p^2 + s^2$ ($qr = ps$)

$= (p-s)^2 =$ R.H.S.

Ex 10) Shraddha deposited Rs. 8000 in a bank which pays annual interest rate of 8%.She kept it with the bank for 10 years with compound interest. Find the total amount she will receive after 10 years . [given $(1.08)^{10} = 2.1589$)]

Solution:

The Amount deposited in a bank is Rs 8000 with 8% compound interest.

Each year, the ratio of the amount to the principal of that year is constant = $\frac{108}{100}$

Hence we get a G.P. of successive amounts.

For P = 8000,

the amount after 1 year is $8000 \times \frac{108}{100}$

the amount after 2 years is $8000 \times \frac{108}{100} \times \frac{108}{100}$

the amount after 3 years is $8000 \times \frac{108}{100} \times \frac{108}{100} \times \frac{108}{100}$.

Therefore after 10 years the amount is

$8000 \left(\frac{108}{100}\right)^{10} = 8000\ (1.08)^{10}$.

$= 8000 \times 2.1589 = 17271$

Thus Shraddha will get Rs 17271 after 10 years.

Ex 11) The number of bacteria in a culture doubles every hour. If there were 50 bacteria originally in the culture, how many bacteria will be there after 5 hours ?

Solution : Given that the number of bacteria doubles every hour .

The ratio of bacteria after 1 hour to that at the begining is 2

after 1 hour = 50 x 2

after 2 hour = 50 x 2^2

after 3 hour = 50 x 2^3

Hence it is a G.P. with a=50 , r = 2 .

To find the number of bacteria present after 5 hours, that is to find t_6 .

$t_6 = ar^5 = 50\ (2)^5 = 50\ (32) = 1600$

ACTIVITIES

Activity 4.1:

Verify whether 1, $\frac{-4}{3}$, $\frac{16}{9}$, $\frac{-64}{27}$, ... is a G.P. If it is a G.P. Find its ninth term.

Solution : Herer $t_1 = 1$, $t_2 = \square$, $t_3 = \frac{16}{9}$,

Consider $\frac{t_2}{t_1} = \square = \frac{-4}{3}$

$\frac{t_4}{t_3} = \frac{\frac{-64}{27}}{\frac{16}{9}} = \square$

Here the ratio is constant. Hence the given sequence is a G.P.

Now $t_9 = \square = ar^8 = 1\left(\frac{-4}{3}\right)^8 = \square$

Activity 4.2:

For a G.P. $a = 3$, $r = 2$, $S_n = 765$, find n.

Solution : $S_n = 765 = \square\ (2^n - 1)$,

$\frac{765}{3} = 255 = 2^n - 1$,

$2^n = \square = 2^8$, n = $\square$

Activity 4.3:

For a G.P. if $t_6 = 486$, $t_3 = 18$, find t_9

Solution : We know that, for a G.P. $t_n = \square$

$\therefore\ t_3 = \square = 18$ ... (I)

$t_6 = \square = 486$ ... (II)

$\therefore\ \frac{t_6}{t_3} = \frac{\square}{\square} = \frac{486}{18}$

$\therefore\ r^{\square} = \square$

$\therefore \quad r = \square$

Now from (I), $t_3 = ar^2 = 18$

$\therefore \quad a = \square$

$\therefore \quad t_9 = ar^8 = \square$

Activity 4.4:

If a, b, c, d are in G.P. then show that (a – b) (b – c) and (c – d) are also in G.P.

Solution : a, b, c, d are in G.P.

$\therefore \quad b^2 = \square$

$\square = bd$

$ad = \square$

To prove that (a – b), $(b - c)$, $(c - d)$ are in G.P.

i.e. to prove that $(b - c)^2 = (a - b)(c - d)$

$\therefore$ RHS $= (a - b)(c - d)$

$= ac - \square - bc + \square$

$= b^2 - bc - bc + \square$

$= b^2 - 2bc + c^2$

$= \square$

$=$ LHS

Activity 4.5:

For a sequence, $S_n = 7(4^n - 1)$, find t_n and show that the sequence is a G.P.

Solution : $S_n = 7(4^n - 1)$

$S_{(n-1)} = 7\square$

$t_n = \square$

$= 7(4^n - 1) - 7(\square)$

$= 7[4^n - 1 - 4^{n-1} + 1]$

$= 7\square$

Activity 4.6:

10 people visited an exibition on the first day. The number of visitors was doubled on the next day and so on. Find i) number of visitors on 9th day. ii) Total number of visitors after 12 days.

Solution : On 1st day number of visitors was $\square$

Number of visitors doubles on next day.

$\therefore$ On 2nd day number of visitors = $\square$

$\therefore$ On 3rd day number of visitors = $\square$

and so on

$\therefore$ Number of visitors are 10, 20, 40, 80, ...

These number forms a G.P. with $a = \square$

$r = \square$

$\therefore$ No. of visitors on 9th day i.e. $t_9 = a.r^{9-1}$

$= \square 2^{\square}$

$= 10 \times 2^{\square}$

$= 10 \times \square =$

Total number of visitors after 12 days

$= S_{12} = a\square$

$= 10\left[\frac{1-2^{12}}{1-2}\right] = \frac{10 \times (1-4096)}{\square}$

$= 10 \times 4095$

$= \square$

Activity 4.7:

Complete the following activity to find sum to n terms of 7+77+777+7777+..............

Let S_n = 7+77+777+7777+........... upto n terms

$= 7\times(\square+\square+\square+\square$........... upto n terms)

$= \frac{7}{9}(\square+\square+\square+\square$........... upto n terms)

$= \frac{7}{9}$ [(10–1)+(–)+(–)+(–) (–)

.... upto n terms)

$= \frac{7}{9}$ [(10+10²+..........+ upto n terms)

– (1+1+.......... upto n terms)]

$= \frac{7}{9}[\square(\ \ )-\square]$ .... using $\left(\frac{a.(r^n-1)}{r-1}\right)$

$$S_n = \frac{7}{9}\left(\frac{\square}{\square}(\ \ ) - n\right) = \square$$

Activity 4.8:

An empty bus arrived at a bus stand. In the first minute two persons boarded the bus. In the second minute 4 persons, in the third minute 8 persons boarded the bus and so on. The bus was full to its seating capacity in 5 minutes. What was the number of seats in the bus?

Solution : In the first minute, 2 persons board.

In the second minute, 4 persons board.

and so on ....

Hence it is a $\square$

with a = $\square$, r = $\square$

The bus was full in 5 minutes

$$S_5 = a\left(\frac{r^5-1}{r-1}\right) = 2\left(\frac{\square - 1}{\square - 1}\right)$$

$$= 2\left(\frac{\square}{\square}\right) = \square$$

The number of seats in the bus = $\square$

Let's remember!

For a G.P. $\{t_n\}$,

1) $\frac{t_{n+1}}{t_n}$ = constant , $\forall\ n \in N$

2) $t_n = a\,r^{n-1}, a \neq 0, r \neq 0$, $\forall\ n \in N$

3) 3 successive terms in G. P. are written as $\frac{a}{r}$, a, ar.

4) 4 successive terms in G. P. are written as $\frac{a}{r^3}$, $\frac{a}{r}$, ar , ar^3. (ratio r^2)

5) 5 successive terms in G. P. are written as $\frac{a}{r^2}$, $\frac{a}{r}$, a , ar , ar^2

EXERCISE 4.1

1) Verify whether the following sequences are G.P. If so, find t_n.

   i) 2,6,18,54,.....................

   ii) 1,-5,25,-125...................

   iii) $\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{5\sqrt{5}}, \frac{1}{25\sqrt{5}}$, ...

   iv) 3, 4, 5, 6,.........

   v) 7, 14, 21, 28,.........

2) For the G.P.

   i) if $r = \frac{1}{3}$, $a = 9$; find t_7

   ii) if $a = \frac{7}{243}$, $r = \frac{1}{3}$ find t_3

   iii) if $a = 7$, $r = -3$ find t_6

   iv) if $a = \frac{2}{3}$, $t_6 = 162$, find r

3) Which term of the G.P. 5,25,125,625,.......is 5^{10}?

4) For what values of x.

   $\frac{4}{3}$, x , $\frac{4}{27}$ are in G.P. ?

5) If for a sequence, $t_n = \frac{5^{n-3}}{2^{n-3}}$, show that the sequence is a G.P.

   Find its first term and the common ratio.

6) Find three numbers in G.P. such that their sum is 21 and sum of their squares is 189.

7) Find four numbers in G.P. such that sum of the middle two numbers is 10/3 and their product is 1.

8) Find five numbers in G. P. such that their product is 1024 and fifth term is square of the third term.

9) The fifth term of a G.P. is x , eighth term of the G.P. is y and eleventh term of the G.P. is z Verify whether $y^2 = x z$.

10) If p,q,r,s are in G.P. show that $p+q$, $q+r$, $r+s$ are also in G.P.

Let's learn.

4.4 Sum of the first n terms of a G.P.

If $\{t_n\}$ is a geometric progression with first term a and common ratio r ; where $a \neq 0$, $r \neq 0$; then the sum of its first n terms is given by

$S_n = a+ar+ar^2+ar^3+ \ldots\ldots\ldots + ar^{n-1} =$

$$a\left(\frac{1-r^n}{1-r}\right), r \neq 1$$

Proof : Consider

$S=1+r+r^2+r^3+\ldots\ldots r^{n-1}$ ……..... (i)

Multiplying both sides by r we get

$r S = r+r^2+r^3+\ldots\ldots r^n$ ……..... (ii)

Subtract (ii) from (i) we get $S - S r = 1-r^n$

$\therefore \quad S(1-r) = 1-r^n$

$$\therefore \quad S = \left(\frac{1-r^n}{1-r}\right), \ldots\ldots\ldots (iii), \quad r \neq 1.$$

Multiplying both sides of equation (i) by a we get ,

$a S = a+ar+ar^2+\ldots\ldots\ldots + ar^{n-1} = S_n$

$$a\left(\frac{1-r^n}{1-r}\right) = S_n \text{ \{from (i) and (iii)\}}$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right), r \neq 1.$$

If we Subtract (i) from (ii) we get,

$$S_n = a\left(\frac{r^n-1}{r-1}\right).$$

Let's Note.

1. $S_n = a. S$
2. If r = 1, then $S_n = n\,a$

Solved Examples

Ex 1) If $a = 1$, $r = 2$ find S_n for the G.P.

Solution : $a = 1$, $r = 2$

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = 1\left(\frac{1-2^n}{1-2}\right) = 2^n - 1 .$$

Ex 2) For a G.P. 0.02,0.04,0.08,0.16,…, find S_n.

Solution : Here a = 0.02, $r = 2$

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = 0.02\left(\frac{1-2^n}{1-2}\right)$$

$$= 0.02 . (2^n -1)$$

Ex 3) For the G.P. 3,–3,3,–3,……… , Find S_n.

Solution :

If n is even , $n = 2k$

$S_{2k} = (3-3) + (3-3) + (3-3) +$ --------- $+ (3-3) = 0$.

If n is odd , $n = 2k +1$

$S_{2k+1} = S_{2k} + t_{2k+1} = 0 + 3 = 3.$

Ex 4) For a G.P. if a=6 , r =2 , find S_{10}.

Solution: $S_n = a\left(\dfrac{1-r^n}{1-r}\right)$,

$$S_{10} = 6\left(\frac{1-2^{10}}{1-2}\right) = 6\left(\frac{1-1024}{-1}\right)$$

$$= 6\left(\frac{-1023}{-1}\right) = 6\,(1023) = 6138.$$

Ex 5) If for a G.P. r=2 , S_{10}=1023, find *a*.

Solution : $S_{10} = a\left(\dfrac{1-2^{10}}{1-2}\right)$

$\therefore$ 1023 = a (1023)

$\therefore$ a =1.

Ex 6) For a G.P. $a = 5$, $r = 2$, $S_n = 5115$, find n .

Solution : $S_n = 5115 = 5\left(\dfrac{2^n-1}{2-1}\right) = 5\,(2^n-1)$,

$$\therefore\ \frac{5115}{5} = 1023 = 2^n - 1$$

$2^n = 1024 = 2^{10}$

$\therefore$ $n = 10$

Ex 7) If for a G.P. $S_3 = 16$, $S_6 = 144$, find the first term and the common ratio of the G.P.

Solution : Given

$S_3 = a\left(\dfrac{1-r^3}{1-r}\right) = 16$ .... (1)

$S_6 = a\left(\dfrac{1-r^6}{1-r}\right) = 144$ ....(2)

Dividing (2) by (1) we get ,

$$\frac{S_6}{S_3} = \frac{r^6-1}{r^3-1} = \frac{144}{16} = 9$$

$$\frac{(r^3-1)(r^3+1)}{(r^3-1)} = 9,$$

$(r^3+1) = 9,$

$r^3 = 8 = 2^3,$

$r = 2$.

Substitute r = 2 in (1) We get

$$a\left(\frac{1-2^3}{1-2}\right) = 16,$$

$$a\left(\frac{1-8}{1-2}\right) = 16,$$

$a\,(7) = 16,$

$$a = \frac{16}{7}$$

Ex 8) Find the sum

9+99+999+9999+..................upto n terms.

Solution : Let S_n = 9+99+999+9999+........... ....upto n terms.

S_n = (10-1) + (100-1) + (1000-1)… to n brackets.

= (10+100+1000+ ...... upto n terms)
– (1+1+1 ...... upto n terms)

Terms in first bracket are in G.P. with a = 10, r = 10 and terms in second bracket are in G.P. with a = r = 1

$$\therefore\ S_n = 10\left(\frac{10^n-1}{10-1}\right) - n$$

$$= \frac{10}{9}(10^n - 1) - n.$$

Ex 9) Find the sum 5+55+555+5555+........ upto n terms.

Solution: Let S_n

= 5+55+555+5555+............ upto n terms.

= 5 (1+11+111+ .... upto n terms)

= $\dfrac{5}{9}$ (9+99+999+ .... upto n terms)

= $\dfrac{5}{9}$ [(10-1) + (100-1) + (1000-1) + ... to n brackets]

= $\dfrac{5}{9}$ [(10 + 100 + 1000 +......... upto n terms)

$-$ (1+1+1+ ………… upto n terms)]

$$= \frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right) - n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$$

Ex 10) Find the sum to n terms

0.3+0.03+0.003+.......... upto n terms

Solution : Let S_n

= 0.3+0.03+0.003+ upto n terms

= 3 [0.1+ +0.01+ 0.001+ ……. ..upto n terms]

$$= 3\left[\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\ldots\ldots\ldots + \frac{1}{10^n}\right]$$

$$= 3 \times a\left(\frac{1-r^n}{1-r}\right) \text{ where } a = \frac{1}{10} \text{ and } r = \frac{1}{10}$$

$$= 3 \times \frac{1}{10}\left(\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}\right)$$

$$= \frac{1}{3}(1 - 0.1^n)$$

Ex 11) Find the n^{th} term of the sequence

0.4 ,0.44,0.444,……………………..

Solution : Here t_1=0.4

t_2=0.44 = 0.4 + 0.04

t_3= 0.444 = 0.4+0.04+0.004

…………………………………

t_n= 0.4 + 0.04 + 0.004 + 0.0004 + ……………….. upto n terms

here t_n is the sum of first n terms of a G.P. with a = 0.4 and r = 0.1

$$t_n = 0.4\left(\frac{1-0.1^n}{1-0.1}\right) = \frac{4}{9}\left[1-(0.1)^n\right]$$

Ex 12) For a sequence , if $S_n = 5(4^n-1)$, find the n^{th} term , hence verify that it is a G.P. ,Also find r .

Solution : $S_n = 5(4^n-1)$, $S_{n-1} = 5(4^{n-1}-1)$

We know that $t_n = S_n - S_{n-1}$

$= 5(4^n - 1) - 5(4^{n-1} - 1)$

$= 5(4^n) - 5 - 5(4^{n-1}) + 5$

$= 5(4^n - 4^{n-1})$

$= 5(4^n - 4^n.4^{-1})$

$= 5(4^n)\left(1 - \frac{1}{4}\right)$

$= 5(4^n)\left(\frac{3}{4}\right)$

$\therefore t_{n+1} = 5(4^{n+1}) \times \frac{3}{4}$

Consider $\frac{t_{n+1}}{t_n} = \frac{5(4^{n+1})}{5(4^n)} = 4$ = constant ,

$\forall\ n \in$ N.

$\therefore r = 4$

$\therefore$ the sequence is a G.P.

Ex 13) Which term of the sequence

$\sqrt{3}$, 3, $3\sqrt{3}$, ... is 243?

Solution : Here a = $\sqrt{3}$, $r = \sqrt{3}$, t_n= 243

$\therefore$ a. r^{n-1} = 243

$$\sqrt{3}\ .\left(\sqrt{3}\right)^{n-1} = 243 = 3^5 = \left(\sqrt{3}\right)^{10}$$

$$\therefore \quad \left(\sqrt{3}\right)^n = \left(\sqrt{3}\right)^{10}$$

$\therefore$ n = 10.

Tenth term of the sequence is 243.

Ex 14) How many terms of G.P.

2 , 2^2, 2^3,2^4.,................ are needed to give the sum 2046.

Solution : Here a=2 , r = 2 , let S_n = 2046.

$\therefore\ 2046 = a\left(\frac{r^n-1}{r-1}\right) = 2\left[\frac{2^n-1}{2-1}\right] = 2\ (2^n-1)$

$1023 = 2^n-1\ ,\ 2^n = 1024\ = 2^{10}\ \therefore\ n = 10$

Ex 15) Mr. Pritesh got the job with an annual salary package of Rs. 400000 with 10% annual increment .Find his salary in the 5 th year and also find his total earnings through salary in 10 years.

[Given $(1.1)^4 = 1.4641$, $(1.1)^{10} = 2.59374$

Solution : In the first year he will get a salary of Rs. 400000 .

He gets an increment of 10% so in the second year his salary will be

$$400000 \times \left(\frac{110}{100}\right) = 440000$$

In the third year his salary will be

$$400000 \times \left(\frac{110}{100}\right)^2 \text{ and so on ....}$$

Hence it is a G.P. with a = 400000 & r = 1.1.

Similarly his salary in the fifth year will be

$$t_5 = ar^4 = 400000\left(\frac{110}{100}\right)^4 = 585640.$$

$$[\because (1.1)^4 = 1.4641]$$

His total income through salary in 10 years

will be $S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$

$$= 400000 \times \left(\frac{2.59374-1}{0.1}\right)$$

$[\because (1.1)^{10} = 2.59374]$

$$= 400000\left(\frac{1.59374}{0.1}\right)$$

$= 400000\ [\ 15.9374\ ] = \ 63{,}74{,}960.$

$\therefore$ Mr. Pritesh will get Rs.5,85,640 in the fifth year and his total earnings through salary in 10 years will be Rs. 63,74,960.

Ex 16) A teacher wanted to reward a student by giving some chocolates. He gave the student two choices. He could either have 50 chocolates at once or he could get 1 chocolate on the first day, 2 on the second day, 4 on the third day and so on for 6 days. Which option should the student choose to get more chocolates?

Ans : We need to find sum of chocolates in 6 days.

According to second option teacher gives 1 chocolate on the first day, 2 on the second day, 4 on the third day, and so on. Hence it is a G. P . with $a = 1$, $r = 2$.

If the number of chocolates collected in this way is greater than 50 we have to assume this is the better way.

By using $S_n = a\left(\frac{r^n-1}{r-1}\right)$

$$= 1\left(\frac{2^6-1}{2-1}\right)$$

$$= 64 - 1 = 63$$

Hence the student should choose the second way to get more chocolates.

EXERCISE 4.2

1) For the following G.P.s, find S_n

 i) 3 , 6 , 12 , 24 , ...

 ii) $p\ ,q\ ,\frac{q^2}{p}\ ,\frac{q^3}{p^2}\ , \ldots\ldots\ldots\ldots$

2) For a G.P. if

 i) $a = 2, r = -\frac{2}{3}$, find S_6

 ii) $S_5 = 1023\ , r = 4$, Find a

3) For a G.P. if

 i) $a = 2\ , r = 3\ ,\ S_n = 242$ find n.

 ii) sum of first 3 terms is 125 and sum of next 3 terms is 27, find the value of r.

4) For a G.P.

i) If $t_3 = 20$, $t_6 = 160$, find S_7

ii) If $t_4 = 16$, $t_9 = 512$, find S_{10}

5) Find the sum to *n* terms

i) 3 + 33 + 333 + 3333 + ...

ii) 8 +88 + 888 + 8888 + ...

6) Find the sum to n terms

i) 0.4 + 0.44 + 0.444 + ...

ii) 0.7 + 0.77+ 0.777 + ...

7) Find the n^{th} term of the sequence

i) 0.5, 0.55, 0.555, ...

ii) 0.2, 0.22, 0.222, ...

8) For a sequence, if $S_n = 2(3^n - 1)$, find the n^{th} term , hence show that the sequence is a G.P.

9) If S,P,R are the sum , product and sum of the reciprocals of *n* terms of a G. P. respectively , then verify that $\left(\frac{S}{R}\right)^n = P^2$.

10) If S_n, S_{2n}, S_{3n} are the sum of n,2n,3n terms of a G.P. respectively , then verify that $S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

4.5 Sum of infinite terms of a G. P.

We have learnt how to find the sum of first *n* terms of a G.P.

If the G.P. is infinite, does it have a finite sum?

Let's understand

Let's find sum to infinity.How can we find it?

We know that for a G.P.

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \left(\frac{a}{1-r}\right)r^n$$

If |r| < 1 then,as n tends to infinity, r^n tends to zero.

Hence S_n tends to $\frac{a}{1-r}$.

(as $\left(\frac{a}{1-r}\right) r^n$ tends to zero)

Hence the sum of an infinite G.P. is given by $\frac{a}{1-r}$, when | r | < 1.

Note : If $|r| \geq 1$ then sum to infinite terms does not exist.

SOLVED EXAMPLES

EX 1) Determine whether the sum of all the terms in the series is finite ?

In case it is finite find it.

i) $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\ldots\ldots\ldots$

ii) $1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \ldots\ldots\ldots\ldots\ldots\ldots$

iii) $-\frac{3}{5}, \frac{-9}{25}, \frac{-27}{125}, \frac{-81}{625} \ldots\ldots\ldots\ldots\ldots$

iv) 1,-3,9,-27,81, ………….............

Solution: i) $r = \frac{1}{3}$

Here $a = \frac{1}{3}$, $r = \frac{1}{3}$, $|r| < 1$

$\therefore$ Sum to infinity exists.

$$S = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \left[\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}\right] = \frac{1}{2}$$

ii) Here $a = 1$, $r = -\frac{1}{2}$, $|r| < 1$

$\therefore$ Sum to infinity exist

$$S = \frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}.$$

iii) Here $a = -\frac{3}{5}$, $r = \frac{3}{5}$, $|r| < 1$

$\therefore$ Sum to infinity exists

$$S = \frac{a}{1-r} = \frac{-\frac{3}{5}}{1-\left(\frac{3}{5}\right)} = \frac{-\left(\frac{3}{5}\right)}{\left(\frac{2}{5}\right)} = \frac{-3}{2}.$$

iv) Here a=1 , r = -3

As $|r| \not< 1$

$\therefore$ Sum to infinity does not exist.

Let's learn.

RECURRING DECIMALS :

We know that every rational number has decimal form.

For example ,

$\frac{7}{6} = 1.166666\text{.........} = 1.1\dot{6}$

$\frac{5}{6} = 0.833333\text{....} = 0.8\dot{3}$

$\frac{-5}{3} = -1.666666 = -1.\dot{6}$

$\frac{22}{7} = 3.142857142857\text{.......} = 3.\overline{142857}$

$\frac{23}{99} = 0.23232323\text{............} = 0.\overline{23}$

We can use G.P. to represent recurring decimals as a rational number .

SOLVED EXAMPLES

Ex i) 0.66666.....

$= 0.6 + 0.06 + 0.006 + \ldots\ldots\ldots$

$= \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \ldots\ldots ,$

the terms are in G.P. with $a = 0.6$, $r = 0.1 < 1$

$\therefore$ Sum to infinity exists and is given by

$$\frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

ii) $0.\overline{46} = 0.46+0.0046+0.000046 + \ldots\ldots$ the terms are in G.P. with a = 0.46 , $r = 0.01 < 1$.

$\therefore$ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.46}{1-(0.01)} = \frac{0.46}{0.99} = \frac{46}{99}.$$

iii) $2.\bar{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots\ldots$ After the first term,the terms are in G.P. with $a = 0.5$, $r = 0.1 < 1$

$\therefore$ Sum to infinity exists

$$= \frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

$\therefore\ 2.\bar{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots\ldots$

$$= 2 + \frac{5}{9} = \frac{23}{9}$$

EXERCISE 4.3

1) Determine whether the sums to infinity of the following G.P.s exist ,if exist find them

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots\ldots\ldots\ldots\ldots$

ii) $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27} \ldots\ldots$

iii) $-3, 1, \frac{-1}{3}, \frac{1}{9}, \ldots\ldots\ldots\ldots\ldots$

iv) $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \frac{16}{5}, \ldots\ldots\ldots\ldots\ldots$

2) Express the following recurring decimals as a rational number.

i) $0.\overline{32}$.

ii) $3.\dot{5}$

iii) $4.\overline{18}$

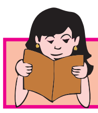
iv) $0.3\overline{45}$

v) $3.4\overline{56}$

3) If the common ratio of a G.P. is $\frac{2}{3}$ and sum of its terms to infinity is 12. Find the first term.

4) If the first term of a G.P. is 16 and sum of its terms to infinity is $\frac{176}{5}$, find the common ratio.

5) The sum of the terms of an infinite G.P. is 5 and the sum of the squares of those terms is 15. Find the G.P.



Let's learn.

Harmonic Progression (H. P.)

Definition : A sequence $t_1, t_2, t_3, t_4, \ldots\ldots\ldots\ldots\ldots t_n$, ($t_n \neq 0$, $n \in$ N) is called a harmonic progression if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots\ldots\ \frac{1}{t_n}, \ldots\ldots$ are in A.P.

For example ,

i) $\frac{1}{7}, \frac{1}{11}, \frac{1}{15} \ldots\ldots$

ii) $\frac{1}{4}, \frac{1}{9}, \frac{1}{14}, \frac{1}{19} \ldots\ldots$

iii) $\frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \frac{1}{14} \ldots\ldots$

iv) $\frac{1}{4}, \frac{3}{14}, \frac{3}{16}, \frac{1}{6} \ldots\ldots\ldots$

SOLVED EXAMPLES

Ex1) Find the n^{th} term of the H.P.

$$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7} \ldots\ldots\ldots$$

Solution : Here 2, $\frac{5}{2}$, 3, $\frac{7}{2}$ .... are in A.P. with a = 2 and d = $\frac{1}{2}$ hence

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \ldots\ldots\ldots$.are in H.P.

For A.P.

$$t_n = a + (n\text{-}1)\, d = 2 + (n\text{-}1)\,\frac{1}{2}$$

$$= 2 + \frac{1}{2}n - \frac{1}{2}$$

$$t_n = \frac{3}{2} + \frac{n}{2} = \frac{3+n}{2}$$

For H.P. $t_n = \frac{2}{3+n}$

Ex2) Find the n^{th} term of H.P. $\frac{1}{5}, 1, \frac{-1}{3}, \frac{-1}{7}, \ldots\ldots\ldots$

Solution: Since 5,1,-3,-7,............... are in A.P.

with a = 5 and d = −4

Hence t_n = a + $(n-1)$ d

= 5 + $(n-1)(-4)$

= 5 −4n +4 = 9−4n.

For H.P. $t_n = \frac{1}{9-4n}$

Activity : 4.9

Find the n^{th} term of the following H.P.

$$\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22} \ldots\ldots\ldots$$

Solution : Here 2, 7, 12, 17, 22 .... are in $\square$

with a = $\square$ and (d) = $\square$

hence $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \frac{1}{17}, \frac{1}{22}, ...$ are in $\square$

$$t_n = a + (n-1)\,d =$$

$$= \square + (n-1)\square$$

$$\therefore\ t_n \text{ of the H.P.} = \square$$

Let's learn.

Types of Means:

Arithmetic mean (A. M.)

If x and y are two numbers, their A.M. is defined by $A = \frac{x+y}{2}$.

We observe that x, A , y form an AP.

Geometric mean (G. M.)

If x and y are two numbers having same sign (positive or negative), their G.M. is defined by

$G = \sqrt{xy}$. We observe that x, G, y form a G P .

Harmonic mean (H. M.)

If x and y are two numbers, their H.M. is defined by $H = \frac{2xy}{x+y}$.

We observe that x, H, y form an HP .

These definitions can be extended to n numbers as follows

$$A = \frac{x_1 + x_2 + x_3 + ...x_n}{n}$$

$$G = \sqrt[n]{x_1.x_2.x_3.......x_n}$$

$$\text{H.M.} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}}$$

Theorem : If A, G and H are A.M., G.M., H.M. of two positive numbers respectively, then

i) $G^2 = AH$ ii) $H < G < A$

Proof : let x and y be the two positive numbers.

$$A = \frac{x+y}{2},\ G = \sqrt{xy},\ H = \frac{2xy}{x+y}$$

$$\text{RHS} = AH = \frac{x+y}{2}\ \frac{2xy}{x+y}$$

$$= xy = G^2 = \text{L.H.S.}$$

$$\text{Consider } A-G = \frac{x+y}{2} - \sqrt{xy}$$

$$= \frac{1}{2}(x+y-2\sqrt{xy})$$

$$A-G = \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 > 0$$

$$\therefore A > G \text{.............................. (I)}$$

$$\therefore \frac{A}{G} > 1 \text{................................ (II)}$$

Now consider $G^2 = AH$

$$\frac{G}{H} = \frac{A}{G} > 1 \qquad \text{(FROM II)}$$

$$\therefore \frac{G}{H} > 1\ \therefore\ G > H \text{ .... (III)}$$

From (I) and (III) $H < G < A$

Note : If $x = y$ then H = G = A

n arithmetic means between a and b :

Let A_1, A_2, A_3, ... be the n A.M.s between a and b ,

then a, A_1, A_2, A_3, ... , A_n , b is an A.P.

Here total number of terms are n+2

$$b = t_{n+2} = a + [\,(n+2) - 1\,]\,d$$

$$b = a + (n+1)\,d$$

$$d = \frac{b-a}{n+1}$$

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + 2\,\frac{b-a}{n+1}$$

$$A_3 = a + 3d = a + 3\,\frac{b-a}{n+1}$$

.

.

$$A_n = a + n\,d$$

$$= a + n\,\frac{b-a}{n+1} = \frac{a(n+1)}{n+1} + n\,\frac{b-a}{n+1}$$

$$= \frac{a\,(n+1) + n\,(b-a)}{n+1}$$

$$A_n = \frac{a+nb}{n+1}\,.$$

n geometric means between a and b :

Let $G_1, G_2, G_3, G_4, \ldots\ldots G_n$ be the n G.M.s between a and b ,

then $a, G_1, G_2, G_3, \ldots\ldots G_n, b$ are in G.P

Here total number of terms are n+2

$$\therefore\; t_{n+2} = b = a\,(r)^{n+1}$$

$$\therefore\; r^{n+1} = \frac{b}{a}$$

$$\therefore\; r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = a\,r^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+2}}\ldots\ldots\ldots\ldots$$

$$G_n = a\,r^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Examples based on means

Ex : 1 Find A.M.,G.M.,H.M. of the numbers 4 and 16

Solution : Let x = 4 and y = 16

$$A = \frac{x+y}{2} = \frac{20}{2} = 10 \quad \therefore A = 10$$

$$G = \sqrt{xy} = \sqrt{64} = 8\ ,\ G=8.\ ,$$

$$H = \frac{2xy}{x+y} = \frac{2(4)(16)}{4+16} = \frac{128}{20} = \frac{32}{5}.$$

Ex2) Insert 4 arithmetic means between 2 and 22.

Solution:

let A_1, A_2, A_3, A_4 be 4 arithmetic means between 2 and 22

$\therefore$ 2, A_1, A_2, A_3, A_4, 22 are in AP with

$a = 2,\quad t_6 = 22,\quad n = 6$.

$\therefore$ 22 = 2 + (6-1)d = 2 + 5d

20 = 5d , d=4

$A_1 = a+d = 2+4 = 6$,

$A_2 = a+2d = 2+2\times4 = 2+8 = 10$,

$A_3 = a+3d = 2+3\times4 = 2+12 = 14$

$A_4 = a+4d = 2 + 4\times4 = 2+16 = 18$.

$\therefore$ the 4 arithmetic means between 2 and 22 are 6,10,14,18.

Ex: 3 Insert two numbers between $\frac{2}{9}$ and $\frac{1}{12}$ so that the resulting sequence is a H.P.

Solution : let the required numbers be $\frac{1}{H_1}$ and $\frac{1}{H_2}$

$\therefore\; \frac{2}{9}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{12}$ are in H.P .

$\frac{9}{2}$, H_1 , H_2, 12 are in A.P.

$$t_1 = a = \frac{9}{2},\ t_4 = 12 = a+3d = \frac{9}{2} + 3d\,.$$

$$3d = 12 - \frac{9}{2} = \frac{24-9}{2} = \frac{15}{2}$$

$$d = \frac{5}{2}$$

$$t_2 = H_1 = a+d = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7.$$

$$t_3 = H_2 = a+2d = \frac{9}{2} + 2\times\frac{5}{2} = \frac{19}{2}\,..$$

For resulting sequence $\frac{1}{7}$ and $\frac{2}{19}$ are to be inserted between $\frac{2}{9}$ and $\frac{1}{12}$

Ex: 4 Insert two numbers between 1 and 27 so that the resulting sequence is a G. P.

Solution: Let the required numbers be G_1 and G_2

$\therefore$ 1, G_1, G_2, 27 are in G.P.

$\therefore$ $t_1 = 1$, $t_2 = G_1$, $t_3 = G_2$, $t_4 = 27$

$\therefore$ $a = 1$, $t_4 = ar^3 = 27$

$\therefore$ $r^3 = 27 = 3^3 \therefore r = 3$.

$t_2 = G_1 = ar = 1 \text{ x } 3 = 3$

$t_3 = G_2 = ar^2 = 1(3)^2 = 9$

$\therefore$ 3 and 9 are the two required numbers.

Ex: 5 The A.M. of two numbers exceeds their G.M. by 2 and their H.M. by 18/5 .Find the numbers.

Solution : Given A= G + 2 $\quad \therefore G = A - 2$

Also $A = H + \frac{18}{5} \quad \therefore H = A - \frac{18}{5}$

We know that $G^2 = AH$

$(A-2)^2 = A\left(A - \frac{18}{5}\right)$

$A^2 - 4A + 2^2 = A^2 - \frac{18}{5}A$

$\frac{18}{5}A - 4A = -4$

$-2A = -4 \times 5, \quad \therefore A = 10$

Also $G = A - 2 = 10 - 2 = 8$

$\therefore$ $A = \frac{x+y}{2} = 10$, $x+y = 20$, $y = 20 - x$ . ...(i)

Now $G = \sqrt{xy} = 8 \quad \therefore xy = 64$

$\therefore$ $x(20 - x) = 64$

$20x - x^2 = 64$

$x^2 - 20x + 64 = 0$

$(x-16)(x-4) = 0$

$x = 16$ or $x = 4$.

$\therefore$ If $x = 16$, then $y = 4$ $\quad \because y = 20 - x$

$\therefore$ If $x = 4$, then $y = 16$.

The required numbers are 4 and 16.

EXERCISE 4.4

1) Verify whether the following sequences are H.P.

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots\ldots\ldots$

ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots\ldots\ldots\ldots$

iii) $\frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \ldots\ldots\ldots\ldots$

2) Find the n^{th} term and hence find the 8^{th} term of the following H.P.s

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots\ldots\ldots\ldots$

ii) $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\ldots\ldots\ldots$

iii) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \ldots\ldots\ldots\ldots$

3) Find A.M. of two positive numbers whose G.M. and H. M. are 4 and $\frac{16}{5}$

4) Find H.M. of two positive numbers whose A.M. and G.M. are $\frac{15}{2}$ and 6

5) Find G.M. of two positive numbers whose A.M. and H.M. are 75 and 48

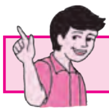
6) Insert two numbers between $\frac{1}{7}$ and $\frac{1}{13}$ so that the resulting sequence is a H.P.

7) Insert two numbers between 1 and −27 so that the resulting sequence is a G.P.

8) Find two numbers whose A.M. exceeds their

G.M. by $\frac{1}{2}$ and their H.M. by $\frac{25}{26}$

9) Find two numbers whose A.M. exceeds G.M. by 7 and their H.M. by $\frac{63}{5}$.



Let's remember!

1) For an A.P. $t_n = a + (n-1)\,d$

2) For a G.P. $t_n = a\,r^{\,n-1.}$

3) A.M. of two numbers A= $\frac{x+y}{2}$

4) G.M. of two numbers G = $\sqrt{xy}$

5) H.M. of two numbers H= $\frac{2xy}{x+y}$

6) $G^2 = AH$

7) If $x = y$ then A = G = H [where x and y are any two numbers]

8) If $x \neq y$ then H < G < A.

4.6 Special Series (sigma Notation)

The symbol $\sum$ (the Greek letter sigma) is used as the summation sign. The sum $a_1+a_2+a_3+a_4+\ldots+a_n$ is expressed as

$\sum_{r=1}^{n} a_r$ (read as sigma a_r , r going from 1 to n)

For example :

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + ... + x_n$$

$$\sum_{i=1}^{9} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9$$

$$\sum_{i=3}^{10} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + x_3^2 + ... + x_n^2$$

$$\sum_{i=1}^{n} x_i y_i = x_1y_1 + x_2y_2 + x_3y_3 + ... + x_ny_n$$

Let's write some important results using $\sum$ notation

Result: 1)

The sum of the first n natural numbers

$$= \sum_{r=1}^{n} r = \frac{n\,(n+1)}{2}$$

Result 2)

The sum of squares of first n natural numbers

$$= \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Result 3)

The sum of the cubes of the first n natural

numbers $= \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

4.6.1 Properties of Sigma Notation

i) $\sum_{r=1}^{n} kt_r = k \sum_{r=1}^{n} t_r$,

Where k is a non zero constant.

ii) $\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$

iii) $\sum_{r=1}^{n} 1 = n$

iv) $\sum_{r=1}^{n} k = k \sum_{r=1}^{n} 1 = k\,n,$

Where k is a non zero constant.

SOLVED EXAMPLES

Ex 1) Evaluate $\sum_{r=1}^{n} (8r-7)$

Solution :

$$\sum_{r=1}^{n} (8r-7) = \sum_{r=1}^{n} 8r - \sum_{r=1}^{n} 7$$

$$= 8 \sum_{r=1}^{n} r - 7 \sum_{r=1}^{n} 1 = 8\left(\frac{n(n+1)}{2}\right) - 7n$$

$= 4(n^2+n) - 7n = 4n^2 + 4n - 7n = 4n^2 - 3n.$

Ex 2) Find $\sum_{r=1}^{17} (3r-5)$

Solution : $\sum_{r=1}^{17} (3r-5) = \sum_{r=1}^{17} 3r - \sum_{r=1}^{17} 5$

$= 3\sum_{r=1}^{17} r - 5 \sum_{r=1}^{17} 1$

$= 3\,\frac{17(17+1)}{2} - 5\,(17)$

$= 3 \times 17 \times \frac{18}{2} - 85$

$= 3 \times 17 \times 9 - 85$

$= 51 \times 9 - 85$

$= 459 - 85$

$= 374.$

Ex 3) Find $3^2 + 4^2 + 5^2 + \text{------} + 29^2$.

Solution: $3^2 + 4^2 + 5^2 + \text{------} + 29^2$

$= (1^2 + 2^2 + 3^2 + \text{----} + 29^2) - (1^2 + 2^2)$

$= \sum_{r=1}^{29} r^2 - \sum_{r=1}^{2} r^2$

$= \frac{29(29+1)(58+1)}{6} - \frac{2(2+1)(4+1)}{6}$

$= 29 \times 30 \times \frac{59}{6} - 2 \times 3 \times \frac{5}{6}$

$= 29 \times 5 \times 59 - 5$

$= 5\,(29 \times 59 - 1) = 5\,(1711 - 1)$

$= 5\,(1710) = 8550$

Ex 4) Find $100^2 - 99^2 + 98^2 - 97^2 + \text{-------} + 2^2 - 1^2$

Solution : $100^2 - 99^2 + 98^2 - 97^2 + \text{-------} + 2^2 - 1^2$

$= (100^2 + 98^2 + 96^2 + \text{-------} + 2^2) -$

$(99^2 + 97^2 + 95^2 + \text{-------} + 1^2)$

$= \sum_{r=1}^{50} (2r)^2 - \sum_{r=1}^{50} (2r-1)^2$

$= \sum_{r=1}^{50} (4r^2 - 4r^2 + 4r - 1)$

$= \sum_{r=1}^{50} (4r - 1)$

$= \sum_{r=1}^{50} 4r - \sum_{r=1}^{50} 1$

$= 4\sum_{r=1}^{50} r - \sum_{r=1}^{50} 1 = 4 \times \frac{50(50+1)}{2} - 50$

$= 2 \times 50 \times 51 - 50$

$= 50\,(2 \times 51 - 1)$

$= 50\,(101)$

$= 5050.$

Ex 5) Find $\sum_{r=1}^{n} \frac{1^2+2^2+3^2+\text{-----------}+r^2}{r+1}$

Solution : $\sum_{r=1}^{n} \frac{1^2+2^2+3^2+\text{-------------}r^2}{r+1}$

$= \sum_{r=1}^{n} \frac{r(r+1)(2r+1)}{6(r+1)}$

$= \frac{1}{6} \sum_{r=1}^{n} (2r^2 + r)$

$= \frac{1}{6}\left(2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$

$= \frac{1}{6}\left[2\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{6}\left[\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}\right]$

$= \frac{n(n+1)(4n+5)}{36}$.

Ex 6) Find $1 \times 5 + 3 \times 7 + 5 \times 9 + 7 \times 11$ .... + upto n terms .

Solution : Consider first factor of each term.1,3,5,7,------ are in A.P. with a =1, d=2.

$t_r = a + (r-1)d = 1 + (r-1)\,2 = 2r - 1.$

Also the second factors 5,7,9,11,----- are in A.P. with $a = 5$, $d = 2$.

$t_r = 5 + (r-1)\,2 = 5 + 2r - 2 = 2r + 3$

$$\therefore \quad S_n = \sum_{r=1}^{n} (2r-1)(2r+3)$$

$$= \sum_{r=1}^{n} (4r^2 + 4r - 3)$$

$$= 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$$

$$= 4\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - 3n$$

$$= n\left[\frac{2(n+1)(2n+1)}{3}\right] + 2n(n+1) - 3n$$

$$= \frac{n}{3}[2(2n^2 + n + 2n + 1) + 6(n+1) - 9]$$

$$= \frac{n}{3}[(4n^2 + 6n + 2) + 6n - 3]$$

$$= \frac{n}{3}(4n^2 + 12n - 1)$$

Ex 7) If $\dfrac{2+4+6+ \text{---} + \textit{upto n terms}}{1+3+5+ \text{---} + \textit{upto n terms}} = \dfrac{20}{19}$,

Find the value of n.

Solution :

Here the terms in the numerator are even numbers hence the general term is $2r$, terms in the denominator are odd numbers hence the general term is $2r - 1$.

$$\therefore \quad \frac{2+4+6+ \text{---} + \textit{upto n terms}}{1+3+5+ \text{---} + \textit{upto n terms}} = \frac{20}{19}$$

$$\frac{\sum_{r=1}^{n} 2r}{\sum_{r=1}^{n}(2r-1)} = \frac{20}{19}$$

$$\frac{2\sum_{r=1}^{n} r}{2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1} = \frac{20}{19}$$

$$2\frac{n(n+1)}{2} \times 19 = 20 \times 2\frac{n(n+1)}{2} - 20 \times n$$

$n(n+1)\ 19 = 20\, n(n+1) - 20n$

dividing by n we get, $19(n+1) = 20(n+1) - 20$

$19n + 19 = 20n + 20 - 20$

$n = 19.$

EXERCISE 4.5

1) Find the sum $\sum_{r=1}^{n} (r+1)(2r-1)$

2) Find $\sum_{r=1}^{n} (3r^2 - 2r + 1)$

3) Find $\sum_{r=1}^{n} \dfrac{1+2+3+ \text{---} + r}{r}$

4) Find $\sum_{r=1}^{n} \dfrac{1^3+2^3+ \text{---} + r^3}{r(r+1)}$

5) Find the sum 5 x 7 + 9 × 11 + 13 × 15 + --- upto n terms.

6) Find the sum $2^2+4^2+6^2+8^2+$ --- upto n terms

7) Find $(70^2 - 69^2) + (68^2 - 67^2) + (66^2 - 65^2) + \text{---} + (2^2 - 1^2)$

8) Find the sum $1\times3\times5 + 3\times5\times7 + 5\times7\times9 + \text{---} (2n-1)(2n+1)(2n+3)$

9) Find n, if

$$\frac{1\times2+2\times3+3\times4+4\times5+ \text{---} + \textit{upto n terms}}{1+2+3+4+ \text{---} + \textit{upto n terms}} = \frac{100}{3}.$$

10) If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that

$9S_2^2 = S_3(1+8S_1)$.

MISCELLANEOUS EXERCISE - 4

1) In a G.P., the fourth term is 48 and the eighth term is 768 . Find the tenth term.

2) For a G.P. $a = \dfrac{4}{3}$ and $t_7 = \dfrac{243}{1024}$, find the value of r.

3) For a sequence , if $t_n = \frac{5^{n-2}}{7^{n-3}}$, verify whether the sequence is a G.P. If it is a G.P. , find its first term and the common ratio.

4) Find three numbers in G.P. such that their sum is 35 and their product is 1000.

5) Find 4 numbers in G.P. such that the sum of middle 2 numbers is 10/3 and their product is 1.

6) Find five numbers in G.P. such that their product is 243 and sum of second and fourth number is 10.

7) For a sequence $S_n = 4(7^n - 1)$ verify whether the sequence is a G.P.

8) Find 2+22+222+2222+ ----------------- upto *n* terms.

9) Find the n^{th} term of the sequence 0.6,0.66,0.666,0.6666,----------

10) Find $\sum_{r=1}^{n} (5r^2 + 4r - 3)$

11) Find $\sum_{r=1}^{n} r(r-3)(r-2)$

12) Find $\sum_{r=1}^{n} \frac{1^2 + 2^2 + 3^2 + -------- + r^2}{2r+1}$

13) Find $\sum_{r=1}^{n} \frac{1^3 + 2^3 + 3^3 + ------- + r^3}{(r+1)^2}$

14) Find 2 × 6 + 4 × 9 + 6 × 12 + ----------- upto n terms.

15) Find $12^2 + 13^2 + 14^2 + 15^2 +$ ----------- 20^2

16) Find $(50^2 - 49^2) + (48^2 - 47^2) + (46^2 - 45^2) +$ ----- $+ (2^2 - 1^2)$.

17) In a G.P. if $t_2 = 7$, $t_4 = 1575$ find r

18) Find k so that $k-1, k, k+2$ are consecutive terms of a G.P.

19) If p^{th} , q^{th} and r^{th} terms of a G.P. are x,y,z respectively, find the value of $x^{q-r} \times y^{r-p} \times z^{p-q}$

❖ ❖ ❖

5. LOCUS AND STRAIGHT LINE

Let's study.

- Locus
- Slope of a line
- Perpendicular and parallel lines
- Angle between interesting lines
- Equations of lines in different forms :
  - ➢ Slope point form
  - ➢ Slope intercept form
  - ➢ Two points form
  - ➢ Double intercept form
  - ➢ Normal form
  - ➢ General form
- Distance of a point from a line
- Distance between parallel lines

Let's recall.

We are familiar with the perpendicular bisector of a segment, the bisector of an angle, circle and triangle etc.

These geometrical figures are sets of points in plane which satisfy certain conditions.

- The perpendicular bisector of a segment in a plane is the set of points in the plane which are equidistant from the end points of the segment. This set is a line.
- The bisector of an angle is the set of points in the plane of the angle which are equidistant from the arms of the angle. This set is a ray.

Let's learn.

5.1 : DEFINITION : LOCUS : A set of points in a plane which satisfy certain geometrical condition (or conditions) is called a locus.

L = {P | P is a point in a plane and P satisfy given geometrical condition}

Here P is the representative of all points in L. L is called the *locus* of point P. Locus is a set of points. Every locus has corresponding geometrical figure. We may say P is a point in the locus or a point on the locus.

Examples:

- The perpendicular bisector of segment AB is the set M = { P | P is a point in a plane such that $PA = PB$ }.
- The bisector of angle AOB is the set :
  D = { P | P is a point in the plane such that P is equidistant from OA and OB }
  = { P | $\angle$ POA = $\angle$ POB }
- The circle with center O and radius 4 is the set L = { P | $OP = 4$, P is a point in the plane }

The plural of locus is loci.

5.2 : EQUATION OF LOCUS : Every point in XY plane has a pair of co–ordinates. If an equation is satisfied by co–ordinates of all points on the locus and if any point whose co–ordinates satisfies the equation is on the locus then that equation is called the equation of the locus.

Ex.1 We know that the y co–ordinate of every point on the X–axis is zero and this is true for points on the X–axis only. Therefore the equation of the X–axis is $y = 0$. Similarly every point on Y-axis will have x-co-ordinate zero and conversely, so its equation will be $x = 0$.

Note: The x-co-ordinate is also called as abscissa and y-co-ordinate is called as ordinate.

Ex.2 Let $L = \{P \mid OP = 4\}$. Find the equation of L.

Solution : L is the locus of points in the plane which are at 4 unit distance from the origin.

Let $P(x, y)$ be any point on the locus L.

As $OP = 4$, $OP^2 = 16$

$\therefore (x - 0)^2 + (y - 0)^2 = 16$

$\therefore x^2 + y^2 = 16$

This is the equation of the locus L.

The locus is seen to be a circle

Ex.3 Find the equation of the locus of points which are equidistant from A(–3, 0) and B(3, 0). Identify the locus.

Solution : Let P(x, y) be any point on the required locus.

P is equidistant from A and B.

$\therefore$ PA = PB

$\therefore PA^2 = PB^2$

$\therefore (x + 3)^2 + (y - 0)^2 = (x - 3)^2 + (y-0)^2$

$\therefore x^2 + 6x + 9 + y^2 = x^2 - 6x + 9 + y^2$

$\therefore 12x = 0$

$\therefore x = 0$. The locus is the Y–axis.

Let's learn.

Shift of Origin : Let OX, OY be the co-ordinate axes. Let O′(h, k) be a point in the plane. Let the origin be shifted to O′. Let O′X′, O′Y′ be the new co–ordinate axes through O′ and parallel to the axes OX and OY respectively.

Let (x, y) be the co–ordinates of P referred to the co–ordinates axes OX, OY and (X, Y) be the co–ordinates of P referred to the co–ordinate axes O′X′, O′Y′. To find relations between (x, y) and (X, Y).

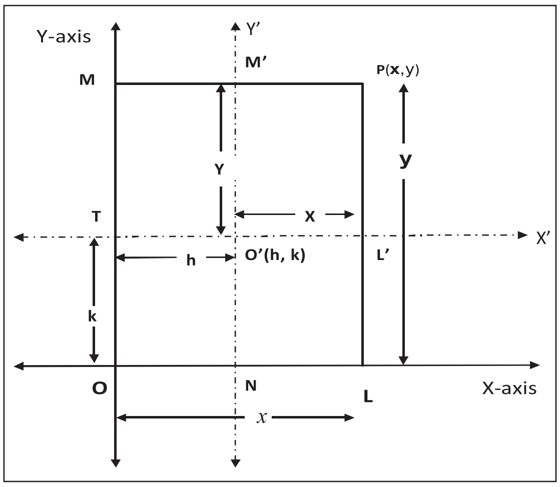


Figure 5.3

Draw $PL \perp OX$ and suppose it intersects $O'X'$ in L'.

Draw $PM \perp OY$ and suppose it intersects $O'Y'$ in M'.

Let $O'Y'$ meet line OX in N and $O'X'$ meet OY in T.

$\therefore$ ON = h, $OT = k$, $OL = x$, $OM = y$,

O′L′ = X, O′M′ = Y

Now $x = OL = ON + NL = ON + O'L'$

$= h + X$

and $y = OM = OT + TM = OT + O'M'$

$= k + Y$

$\therefore$ $\boxed{x = X + h,}$ $\boxed{y = Y + k}$

These equations are known as the formulae for shift of origin.

Ex.4 If the origin is shifted to the point $O'(3,2)$ the directions of the axes remaining the same, find the new co–ordinates of the points (a)A(4, 6) (b) B(2,–5).

Solution : We have $(h, k) = (3,2)$

$x = X + h, y = Y + k$

$\therefore$ $x = X + 3.$ and $y = Y + 2$ .... (1)

(a) $(x, y) = (4, 6)$

$\therefore$ From (1), we get $4 = X + 3$, $6 = Y + 2$

$\therefore$ $X = 1$ and $Y = 4$.

New co–ordinates of A are (1, 4)

(ii) $(x, y) = (2, -5)$

from (1), we get $2 = X + 3$, $-5 = Y + 2$

$\therefore$ $X = -1$ and $Y = -7$.

New co–ordinates of B are $(-1, -7)$

Ex.5 The origin is shifted to the point (-2, 1), the axes being parallel to the original axes. If the new co–ordinates of point A are (7, –4), find the old co–ordinates of point A.

Solution : We have $(h, k) = (-2, 1)$

$x = X + h, y = Y + k$

$\therefore$ $x = X - 2, y = Y + 1$

$(X, Y) = (7, -4)$

we get $x = 7 - 2 = 5$,

$y = -4 + 1 = -3$.

$\therefore$ Old co-ordinates A are (5, -3)

Ex.6 Obtain the new equation of the locus $x^2 - xy - 2y^2 - x + 4y + 2 = 0$ when the origin is shified to (2, 3), the directions of the axes remaining the same.

Solution : Here $(h, k) = (2, 3)$

$\therefore$ $x = X + h, \quad y = Y + k$ gives

$\therefore$ $x = X + 2, \quad y = Y + 3$

The given equation

$x^2 - xy - 2y^2 - x + 4y + 2 = 0$ becomes

$(X+2)^2 - (X + 2)(Y + 3) - 2(Y + 3)^2$

$- (X + 2) + 4(Y + 3) + 2 = 0$

$\therefore$ $X^2 - XY - 2Y^2 - 10Y - 8 = 0$

This is the new equation of the given locus.

EXERCISE 5.1

1. If A(1,3) and B(2,1) are points, find the equation of the locus of point P such that PA = PB.

2. A(−5, 2) and B(4, 1). Find the equation of the locus of point P, which is equidistant from A and B.

3. If A(2, 0) and B(0, 3) are two points, find the equation of the locus of point P such that AP = 2BP.

4. If A(4, 1) and B(5, 4), find the equation of the locus of point P if $PA^2 = 3PB^2$.

5. A(2, 4) and B(5, 8), find the equation of the locus of point P such that

   $PA^2 - PB^2 = 13$.

6. A(1, 6) and B(3, 5), find the equation of the locus of point P such that segment AB subtends right angle at P. ($\angle$ APB = 90°)

7. If the origin is shifted to the point O′(2, 3), the axes remaining parallel to the original axes, find the new co-ordinates of the points

   (a) A(1, 3) (b) B(2, 5)

8. If the origin is shifted to the point O′(1,3) the axes remaining parallel to the original axes, find the old co-ordinates of the points

   (a) C(5, 4) (b) D(3, 3)

9. If the co-ordinates (5,14) change to (8,3) by shift of origin, find the co-ordinates of the point where the origin is shifted.

10. Obtain the new equations of the following loci if the origin is shifted to the point O'(2, 2), the direction of axes remaining the same :

    (a) $3x - y + 2 = 0$

    (b) $x^2 + y^2 - 3x = 7$

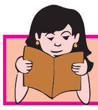
    (c) $xy - 2x - 2y + 4 = 0$

5.3 LINE :

The aim of this chapter is to study a line and its equation. The locus of a point in a plane such that the segment joining any two points on the locus lies completely on the locus is called a line.

The simplest locus in a plane is a line. The characteristic property of this locus is that if we find the slope of a segment joining any two points on this locus, then the slope is constant.

Further we know that if a line meets the X-axis in the point A $(a, 0)$, then a is called the x-intercept of the line. If it meets the Y-axis in the point B $(0, b)$ then b is called the y-intercept of the line.



Let's learn.

5.3.1 : Inclination of a line : The smallest angle made by a line with the positive direction of the X–axis measured in anticlockwise sense is called the inclination of the line. We denote inclination by θ. Clearly $0° < \theta < 180°$.

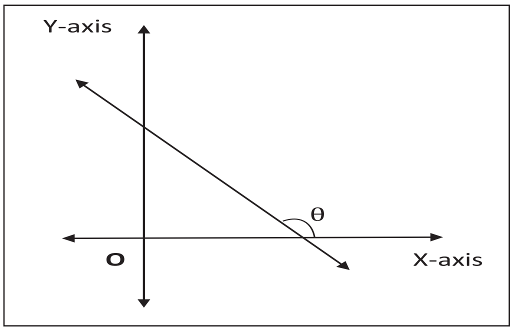


Figure 5.1

Remark : Two lines are parallel if and only if they have the same inclination.

The inclination of the X-axis and a line parallel to the X-axis is Zero. The inclination of the Y-axis and a line parallel to the Y-axis is 90^0.

5.3.2 : Slope of a line : If θ is the inclination of a line then tanθ (if it exists) is called the slope of the line.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ be any two points on a non–vertical line whose inclination is θ then verify that

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_1 \neq x_2.$$

The slope of the Y-axis is not defined. Similarly the slope of a line parallel to the Y–axis is not defined. The slope of the X–axis is 0. The slope of a line parallel to the X–axis is also 0.

Remark : Two lines are parallel if and only if they have the same slope.

Ex.1 Find the slope of the line whose inclination is 60^0.

Solution : The tangent ratio of the inclination of a line is called the slope of the line.

Inclination $\theta = 60°$.

$\therefore$ slope $= \tan\theta = \tan 60° = \sqrt{3}$.

Ex.2 Find the slope of the line which passes through the points A(2, 4) and B(5, 7).

Solution : The slope of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the line AB $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7-4}{5-2} = 1$

Note that $x_1 \neq x_2$.

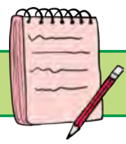
Ex.3 Find the slope of the line which passes through the origin and the point A(-4, 4).

Solution : The slope of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here A(-4, 4) and O(0, 0).

Slope of the line OA $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-4}{0+4} = -1$.

Note that $x_1 \neq x_2$.



Let's study.

5.3.3 : Perpendicular Lines : Lines having slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 \times m_2 = -1$.

Ex.1 Show that line AB is perpendicular to line BC, where A(1, 2), B(2, 4) and C(0, 5).

Solution : Let slopes of lines AB and BC be m_1 and m_2 respectively.

$\therefore m_1 = \dfrac{4-2}{2-1} = 2$ and

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{0-2} = -\frac{1}{2}$$

Now $m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$

$\therefore$ Line AB is perpendicular to line BC.

Ex.2 A(1,2), B(2,3) and C(–2,5) are the vertices of a Δ ABC. Find the slope of the altitude drawn from A.

Solution : The slope of line BC is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{-2-2} = -\frac{2}{4} = -\frac{1}{2}$$

Altitude drawn from A is perpendicular to BC.

If m_2 is the slope of the altitude from A then $m_1 \times m_2 = -1$.

$$\therefore m_2 = \frac{-1}{m_1} = 2.$$

The slope of the altitude drawn from A is 2.

5.3.4 : Angle between intersecting lines :

If θ is the acute angle between non–vertical lines having slopes m_1 and m_2 then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|, \text{ where } 1 + m_1 m_2 \neq 0$$

Ex.1 Find the acute angle between lines having slopes 3 and –2.

Solution : Let $m_1 = 3$ and $m_2 = -2$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3-(-2)}{1+(3)(-2)}\right| = \left|\frac{5}{-5}\right| = 1$$

$\therefore \theta = 45°$

The acute angle between lines having slopes 3 and –2 is 45°.

Ex.2 If the angle between two lines is 45° and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution : If θ is the acute angle between lines having slopes m_1 and m_2 then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Given $\theta = 45°$.

Let $m_1 = \frac{1}{2}$. Let m_2 be slope of the other line.

$$\tan 45° = \left|\frac{\frac{1}{2} - m_2}{1 + \left(\frac{1}{2}\right)m_2}\right| \qquad \therefore 1 = \left|\frac{1-2m_2}{2+m_2}\right|$$

$$\therefore \frac{1-2m_2}{2+m_2} = 1 \text{ or } \frac{1-2m_2}{2+m_2} = -1$$

$\therefore$ $1-2m_2 = 2+m_2$ or $1-2m_2 = -2-m_2$

$\therefore$ $3m_2 = -1$ or $m_2 = 3$

$\therefore$ $m_2 = 3$ or $-\frac{1}{3}$

EXERCISE 5.2

1. Find the slope of each of the following lines which pass through the points :

   (a) (2, –1), (4, 3) (b) (–2, 3), (5, 7)

   (c) (2, 3), (2, –1) (d) (7, 1), (–3, 1)

2. If the X and Y–intercepts of line L are 2 and 3 respectively then find the slope of line L.

3. Find the slope of the line whose inclination is 30^0.

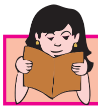
4. Find the slope of the line whose inclination is 45^0.

5. A line makes intercepts 3 and 3 on co–ordiante axes. Find the inclination of the line.

6. Without using Pythagoras theorem show that points A(4,4), B(3, 5) and C(–1, –1) are the vertices of a right angled triangle.

7. Find the slope of the line which makes angle of 45° with the positive direction of the Y–axis measured clockwise.

8. Find the value of k for which points P(k,–1),Q(2,1) and R(4,5) are collinear.



Let's learn.

5.4: EQUATIONS OF LINES IN DIFFERENT FORMS :

We know that line is a locus. Every locus has an equation. An equation in x and y which is satisfied by the co–ordinates of all points on a line and which is not satisfied by the co–ordinates of any point which does not lie on the line is called the equation of the line.

The equation of any line parallel to the Y-axis is of the type $x = k$ (where k is a constant) and the equation of any line parallel to the X-axis is of the type $y = k$. This is all about vertical and horizontal lines.

Let us obtain equations of non–vertical and non –horizontal lines in different forms:

5.4.1 : Slope-Point Form : To find the equation of a line having slope m and which passes through the point $A(x_1, y_1)$.

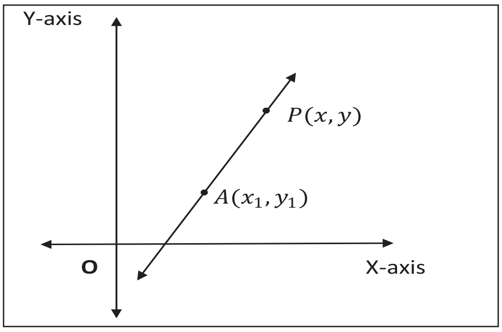


Figure 5.9

The equation of the line having slope m and passing through $A(x_1, y_1)$ is $(y-y_1) = m\ (x-x_1)$.

Remark : In particular if the line passes through the origin O(0,0) and has slope m, then its equation is $y - 0 = m\ (x - 0)$

i.e. $y = mx$

Ex.1) Find the equation of the line passing through the point A(2, 1) and having slope –3.

Solution : Given line passes through the point A(2, 1) and slope of the line is –3.

The equation of the line having slope m and passing through $A(x_1, y_1)$ is $(y-y_1) = m(x-x_1)$.

$\therefore$ the equation of the required line is

$y - 1 = -3(x - 2)$

$\therefore\ y - 1 = -3x + 6$

$\therefore\ 3x + y - 7 = 0$

5.4.2 : Slope-Intercept form : The equation of a line having slope m and which makes intercept c on the Y–axis is y = mx + c.

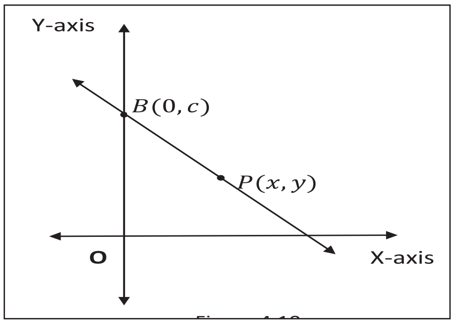


Figure 5.10

Ex.2 Obtain the equation of the line having slope 3 and which makes intercept 4 on the Y–axis.

Solution: The equation of line having slope m and which makes intercept c on the Y–axis is

$y = mx + c.$

$\therefore$ the equation of the line giving slope 3 and making y–intercept 4 is $y = 3x + 4$.

5.4.3 : Two-points Form : The equation of a line which passes through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}$

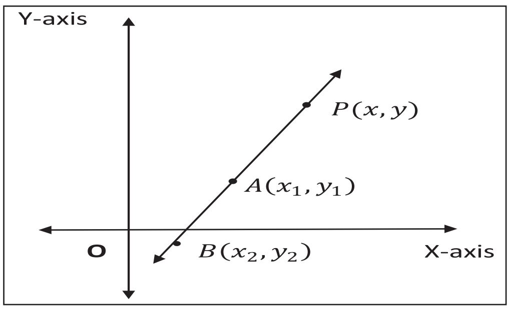


Figure 5.11

Ex.3 Obtain the equation of the line passing through points A(2, 1) and B(1, 2).

Solution : The equation of the line which passes through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}.$$

$\therefore$ The equation of the line passing through points A(2, 1) and B(1, 2) is $\frac{x-2}{1-2}=\frac{y-1}{2-1}$

$$\therefore \frac{x-2}{-1}=\frac{y-1}{1}$$

$$\therefore x-2=-y+1$$

$$\therefore x+y-3=0$$

5.4.4 : Double-Intercept form : The equation of the line which makes non–zero intercepts a and b on the X and Y axes respectively is

$$\frac{x}{a}+\frac{y}{b}=1$$

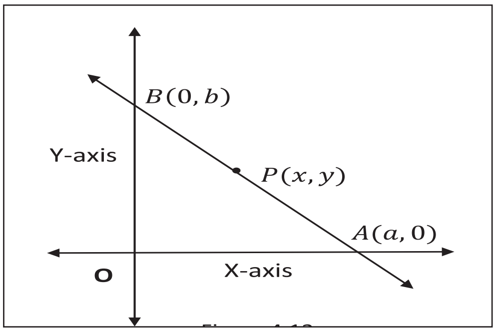


Figure 5.12

Ex.1 Obtain the equation of the line which makes intercepts 3 and 4 on the X and Y axes respectively.

Solution : The equation of the line which makes intercepts a and b on the X and Y co–ordiante axes $\frac{x}{a}+\frac{y}{b}=1$

The equation of the line which makes intercepts 3 and 4 on the co–ordiante axes is $\frac{x}{3}+\frac{y}{4}=1$

$$\therefore 4x+3y-12=0.$$

5.4.5 : Normal Form : Let L be a line and segment ON be the perpendicular (normal) drawn from the origin to line L.

If ON = p. Let the ray ON make angle α with the positive X–axis.

$\therefore$ N = $(p\cos\alpha, p\sin\alpha)$

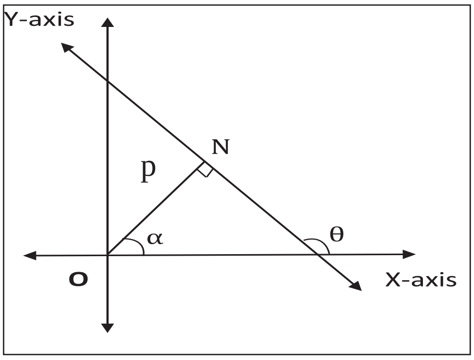


Figure 5.13

Let θ be the inclination of the line L. The equation of the line, the normal to which from the origin has length p and the normal makes angle α with the positive directions of the X–axis, is $x\cos\alpha + y\sin\alpha = p$.

Ex.1 The perpendicular drawn from the origin to a line has length 5 and the perpendicular makes angle 30^0 with the positive direction of the X–axis. Find the equation of the line.

Solution : The perpendicular (normal) drawn from the origin to the line has length 5.

$$\therefore p = 5$$

The perpendicular (normal) makes angle 30^0 with the positive direction of the X–axis.

$$\therefore \theta = 30^0$$

The equation of the required line is

$$x\cos\alpha + y\sin\alpha = p$$

$$\therefore x\cos 30^0 + y\sin 30^0 = p$$

$$\therefore \quad \frac{\sqrt{3}}{2}x+\frac{1}{2}y=5$$

$$\therefore \sqrt{3}x+y-10=0$$

Ex.2 Reduce the equation $\sqrt{3}x-y-2=0$ into normal form. Find the values of p and α.

Solution : Comparing $\sqrt{3}x-y-2=0$ with $ax+by+c=0$ we get $a=\sqrt{3}$, $b=-1$ and $c=-2$.

$$\sqrt{a^2+b^2}=\sqrt{3+1}=2$$

Divide the given equation by 2.

$$\frac{\sqrt{3}}{2}x-\frac{1}{2}y=1$$

$\therefore$ cos 30^0 x – sin 30^0 y = 1 is the required normal form of the given equation.

p = 1 and θ = 30^0

SOLVED EXAMPLES

Find the equation of the line :

(i) parallel to the X–axis and 3 unit below it,

(ii) passing through the origin and having inclination 30^0

(iii) passing through the point $A(5,2)$ and having slope 6.

(iv) passing through the points $A(2–1)$ and $B(5,1)$

(v) having slope $-\frac{3}{4}$ and y–intercept 5,

(vi) making intercepts 3 and 6 on the X and Y axes respectively.

Solution :

(i) Equation of a line parallel to the X–axis is of the form $y=k$,

$\therefore$ the equation of the required line is $y = -3$

(ii) Equation of a line through the origin and having slope m is of the form $y = mx$.

Here, m = tan 30^0 = $\frac{1}{\sqrt{3}}$.

$\therefore$ the equation of the required line is

$$y = \frac{1}{\sqrt{3}}\ x$$

(iii) By using the point–slope form is

$y-y_1$ = m $(x-x_1)$

equation of the required line is

$(y-2)$ = 6 $(x-5)$

i.e. $6x - y - 28 = 0$

(iv) Here (x_1, y_1) = (2–1); (x_2, y_2) = (5,1)

By using the two points form

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

the equation of the required line is

$$\frac{x-2}{5-2}=\frac{y+1}{1+1}$$

$$\therefore\ 2(x-2)=3(y+1)$$

$\therefore$ 2x – 3y – 7 = 0

(v) Given $m=-\frac{3}{4}$, $c=5$

By using the slope intercept form $y=mx+c$

the equation of the required line is

$$y=-\frac{3}{4}x+5 \qquad \therefore\ 3x+4y-20=0$$

(vi) x –intercept = a = 3;

y–intercept = b = 6.

By using the double intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$

the equation of the required line is

$$\frac{x}{3}+\frac{y}{6}=1$$

$$2x+y-6=0$$

EXERCISE 5.3

1. Write the equation of the line :
   a) parallel to the X–axis and at a distance of 5 unit form it and above it.
   b) parallel to the Y– axis and at a distance of 5 unit form it and to the left of it.
   c) parallel to the X– axis and at a distance of 4 unit form the point (–2, 3).
2. Obtain the equation of the line :
   a) parallel to the X–axis and making an intercept of 3 unit on the Y–axis.
   b) parallel to the Y–axis and making an intercept of 4 unit on the X–axis.
3. Obtain the equation of the line containing the point :
   a) $A(2,-3)$ and parallel to the Y–axis.
   b) $B(4,-3)$ and parallel to the X–axis.
4. Find the equation of the line passing through the points $A(2,0)$ and $B(3,4)$.
5. Line $y=mx+c$ passes through points A(2,1) and B(3, 2). Determine m and C.
6. The vertices of a triangle are $A(3,4)$, $B(2,0)$ and $C(1,6)$ Find the equations of
   (a) side BC (b) the median AD
   (c) the line passing through the mid points of sides AB and BC.
7. Find the X and Y intercepts of the following lines :
   (a) $\frac{x}{3}+\frac{y}{2}=1$ (b) $\frac{3x}{2}+\frac{2y}{3}=1$
   (c) $2x-3y+12=0$
8. Find the equations of the lines containing the point $A(3,4)$ and making equal intercepts on the co–ordinates axes.
9. Find the equations of the altitudes of the triangle whose vertices are $A(2, 5)$, $B(6,-1)$ and $C(-4, -3)$.

5.5 : General form of equation of line: We can write equation of every line in the form $ax+by+c=0$. where a, b, c∈R and all are not simultaneously zero.

This form of equation of a line is called the general form.

The general form of $y=3x+2$ is $3x-y+2=0$

The general form of $\frac{x}{2}+\frac{y}{3}=1$ is $3x+2y-6=0$

The slope of the line $ax+by+c=0$ is $-\frac{a}{b}$.

the x–intercept is $-\frac{c}{a}$ and

the y–intercept is $-\frac{c}{b}$.

Ex.1 Find the slope and intercepts made by the following lines :

(a) $x+y+10=0$ (b) $2x+y+30=0$

(c) $x + 3y - 15 = 0$

Solution:

(a) Comparing equation $x+y+10=0$

with $ax + by + c = 0$,

we get $a = 1$, $b = 1$, $c = 10$

$\therefore$ Slope of this line $= -\frac{a}{b} = -1$

The X–intercept is $-\frac{c}{a} = -\frac{10}{1} = -10$

The Y–intercept is $= -\frac{c}{b} = -\frac{10}{1} = -10$

(b) Comparing the equation $2x+y+30=0$

with $ax+by+c=0$.

we get $a = 2$, $b = 1$, $c = 30$

$\therefore$ Slope of this line $= -\frac{a}{b} = -2$

The X–intercept is $-\frac{c}{a} = -\frac{30}{2} = -15$

The Y–intercept is $-\frac{c}{b} = -\frac{30}{1} = -30$

(c) Comparing equation $x + 3y - 15 = 0$ with

$ax + by + c = 0$.

we get $a = 1$, $b = 3$, $c = -15$

$\therefore$ Slope of this line $= -\frac{a}{b} = -\frac{1}{3}$

The x– intercept is $-\frac{c}{a} = -\frac{-15}{1} = 15$

The y– intercept is $-\frac{c}{b} = -\frac{-15}{3} = 5$

Ex.2 Find the acute angle between the following pairs of lines :

a) $12x-4y=5$ and $4x+2y=7$

b) $y=2x+3$ and $y=3x+7$

Solution :

(a) Slopes of lines $12x-4y=5$ and $4x+2y=7$ are $m_1 = 3$ and $m_2 = -2$.

If θ is the acute angle between lines having slope m_1 and m_2 then

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\therefore \tan\theta = \left|\frac{3-(-2)}{1+(3)(-2)}\right| = \left|\frac{5}{-5}\right| = 1$$

$\therefore \tan\theta = 1$ $\qquad$ $\therefore \theta = 45^0$

(b) Slopes of lines $y=2x+3$ and $y=3x+7$ are $m_1 = 2$ and $m_2 = 3$

The acute angle θ between lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\therefore \tan\theta = \left|\frac{2-3}{1+(2)(3)}\right| = \left|\frac{-1}{7}\right| = \frac{1}{7}$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{7}\right).$$

Ex.3 Find the acute angle between the lines $y-\sqrt{3}x+1=0$ and $\sqrt{3}y-x+7=0$.

Solution : Slopes of the given lines are $m_1=\sqrt{3}$ and $m_2=\frac{1}{\sqrt{3}}$.

The acute angle θ between lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\therefore \tan\theta = \left|\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+\sqrt{3}\times\frac{1}{\sqrt{3}}}\right| = \left|\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+1}\right| = \left|\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{2}\right|$$

$$=\left|\frac{1-3}{2\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$$

$$\left|\frac{2}{2\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$$

$\therefore\ \theta = 30^0$

Ex.4 Show that following pairs of lines are perpendicular to each other.

a) $2x-4y=5$ and $2x+y=17$.

b) $y=2x+23$ and $2x+4y=27$

Solution :

(i) Slopes of lines $2x-4y=5$ and $2x+y=17$ are $m_1=\frac{1}{2}$ and $m_2=-2$

Since $m_1.m_2 = \frac{1}{2}\times(-2) = -1$,

given lines are perpendicular to each other.

(ii) Slopes of lines $y = 2x + 23$ and $2x+4y = 27$ are $m_1=-\frac{1}{2}$ and $m_2= 2$.

Since $m_1.m_2 = -\frac{1}{2}\times(2) = -1$, given lines are perpendicular to each other.

Ex.5 Find equations of lines which pass through the origin and make an angle of 45^0 with the line $3x - y = 6$.

Solution : Slope of the line $3x - y = 6$. is 3. Let m be the slope of one of the required line. The angle between these lines is 45^0.

$$\therefore\ \tan 45^0 = \left|\frac{m-3}{1+(m)(3)}\right|$$

$$\therefore\ 1 = \left|\frac{m-3}{1+3m}\right| \qquad \therefore\ |1+3m|=|m-3|$$

$$\therefore\ 1+3m=m-3 \quad \text{or} \quad 1+3m=-(m-3)$$

$\therefore\ m=-2$ or $\frac{1}{2}$

Slopes of required lines are $m_1 = -2$ and $m_2 = \frac{1}{2}$

Required lines pass through the origin.

$\therefore$ Their equations are $y=-2x$ and $y=\frac{1}{2}x$

$\therefore\ 2x+y=0$ and $x-2y=0$

Ex.6 A line is parallel to the line $2x+y=7$ and passes through the origin. Find its equation.

Solution : Slope of the line $2x+y=7$ is -2.

$\therefore$ slope of the required line is also -2

Required line passes through the origin.

$\therefore$ It's equation is $y=-2x$

$\therefore\ 2x+y=0$.

Ex.7 A line is parallel to the line $x+3y=9$ and passes through the point $A(2,7)$. Find its equation.

Solution : Slope of the line $x+3y=9$ is $-\frac{1}{3}$

$\therefore$ slope of the required line is $-\frac{1}{3}$

Required line passes through the point A(2,7).

$\therefore$ It's equation is given by the formula

$(y - y_1) = m\ (x - x_1)$

$\therefore\ (y-7)=-\frac{1}{3}(x-2)$

$\therefore\ 3y - 21 = - x + 2$

$\therefore\ x+3y=23$.

Ex.8 A line is perpendicular to the line $3x+2y-1=0$ and passes through the point $A(1,1)$. Find its equation.

Solution : Slope of the line $3x+2y-1$ is $-\frac{3}{2}$

Required line is perpendicular it.

The slope of the required line is $\frac{2}{3}$.

Required line passes through the point $A(1,1)$.

$\therefore$ It's equation is given by the formula

$(y-y_1)=m(x-x_1)$

$\therefore \quad (y-1)=\frac{2}{3}(x-1)$

$\therefore \quad 3y-3=2x-2$

$\therefore \quad 2x-3y+1=0$.

5.5.1 : Point of intersection of lines : The co–ordinates of the point of intersection of two intersecting lines can be obtained by solving their equations simultaneously.

SOLVED EXAMPLES

Ex.1 Find the co–ordinates of the point of intersection of lines $x+2y=3$ and $2x-y=1$.

Solution : Solving equations $x+2y=3$ and $2x - y = 1$ simultaneously, we get $x=1$ and $y=1$.

$\therefore$ Given lines intersect in point (1, 1)

Ex.2 Find the equation of line which is parallel to the X–axis and which passes through the point of intersection of lines $x+2y=6$ and $2x-y=2$

Solution : Solving equations $x+2y=6$ and $2x-y=2$ simultaneously, we get $x = 2$ and $y = 2$.

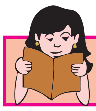
$\therefore$ The required line passes through the point (2, 2) and has slope 0.

(As it is parallel to the X–axis)

$\therefore$ Its equation is given by $(y-y_1) = m(x - x_1)$

$(y-2) = 0(x-2)$

$\therefore \quad y = 2$.



Let's learn.

5.5.2 : The distance of the Origin from a Line :

The distance of the origin from the line $ax+by+c=0$ is given by $p=\left|\frac{c}{\sqrt{a^2+b^2}}\right|$

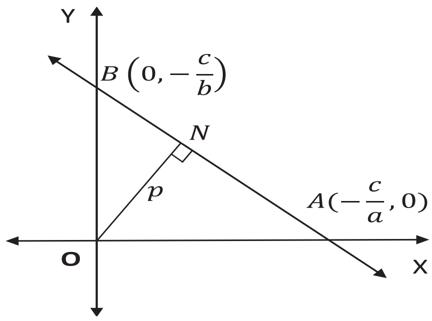


Figure 5.14

5.5.3 : The distance of the point (x_1, y_1) from a line: The distance of the point P (x_1, y_1) from line $ax+by+c=0$ is given by

$p=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

5.5.4 : The distance between two parallel lines :

The distance between two parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is given

by $p=\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$

SOLVED EXAMPLES

Ex.1 Find the distance of the origin from the line $3x+4y+15=0$

Solution : The distance of the origin from the line $ax + by + c = 0$ is given by

$$p=\left|\frac{c}{\sqrt{a^2+b^2}}\right|$$

$\therefore$ The distance of the origin from the line $3x+4y+15=0$ is given by

$$p=\left|\frac{15}{\sqrt{3^2+4^2}}\right| = \frac{15}{5} = 3$$

Ex.2 Find the distance of the point $P(2,5)$ from the line $3x+4y+14=0$

Solution : The distance of the point $P(x_1,y_1)$ from the line $ax+by+c=0$ is given by

$$p=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

$\therefore$ The distance of the point $P(2,5)$ from the line $3x+4y+14=0$ is given by

$$p=\left|\frac{3(2)+4(5)+14}{\sqrt{3^2+4^2}}\right|=\frac{40}{5}=8$$

Ex.3 Find the distance between the parallel lines $6x+8y+21=0$ and $3x+4y+7=0$.

Solution : We write equation $3x+4y+7=0$ as $6x+8y+14=0$ in order to make the coefficients of x and coefficients of y in both equations to be same.

Now by using formula $p=\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$

We get the distance between the given parallel lines as

$$p=\left|\frac{21-14}{\sqrt{6^2+8^2}}\right|=\frac{7}{10}$$

Let's remember!

- **Locus :** A set of points in a plane which satisfy certain geometrical condition (or conditions) is called a locus.
- **Equation of Locus :** Every point in XY plane has Cartesian co–ordinates. An equation which is satisfied by co–ordinates of all points on the locus and which is not satisfied by the co–ordinates of any point which does not lie on the locus is called the equation of the locus.
- **Inclination of a line :** The smallest angle θ made by a line with the positive direction of the X–axis, measured in anticlockwise sense, is called the inclination of the line. Clearly $0^0 < \theta < 180^0$.
- **Slope of a line :** If θ is the inclination of a line then tanθ (if it exists) is called the slope of the line.

  If $A(x_1,y_1)$, $B(x_2, y_2)$ are any two points on the line whose inclination is θ then

  $\tan\theta = \frac{y_2-y_1}{x_2-x_1}$ (if $x_1 \neq x_2$)
- **Perpendicular and parallel lines :** Non-vertical lines having slopes m_1 and m_2 are **perpendicular** to each other if and only if $m_1m_2 = -1$.

Two lines are **parallel** if and only if they have the same slope, that is $m_1 = m_2$.

- **Angle between intersecting lines :** If θ is the acute angle between lines having slopes m_1 and m_2 then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

provided $m_1 m_2 + 1 \neq 0$.

- **Equations of line in different forms :**
- Slope point form : $(y - y_1) = m(x - x_1)$
- Slope intercept form : $y = mx + c$
- Two points form : $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$
- Double intercept form : $\frac{x}{a} + \frac{y}{b} = 1$
- Normal form : $x\cos\alpha + y\sin\alpha = p$
- General form : $ax + by + c = 0$
- **Distance of a point from a line :**
- The distance of the origin from the line

$ax + by + c = 0$ is given by $p = \left|\frac{c}{\sqrt{a^2 + b^2}}\right|$

- The distance of the point $P(x_1, y_1)$ from line $ax + by + c = 0$ is given by

$$p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

- **The distance between the Parallel lines:** The distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$

is give by $p = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

EXERCISE 5.4

1) Find the slope, x–intercept, y–intercept of each of the following lines.

a) $2x + 3y - 6 = 0$

b) $x + 2y = 0$

2) Write each of the following equations in $ax + by + c = 0$ form.

a) $y = 2x - 4$ b) $y = 4$

c) $\frac{x}{2} + \frac{y}{4} = 1$ d) $\frac{x}{3} = \frac{y}{2}$

3) Show that lines $x - 2y - 7 = 0$ and $2x - 4y + 5 = 0$ are parallel to each other.

4) If the line $3x + 4y = p$ makes a triangle of area 24 square unit with the co–ordinate axes then find the value of p.

5) Find the co–ordinates of the circumcenter of the triangle whose vertices are $A(-2, 3), B(6, -1), C(4, 3)$.

6) Find the equation of the line whose X–intercept is 3 and which is perpendicular to the line $3x - y + 23 = 0$.

7) Find the distance of the point $A(-2, 3)$ from the line $12x - 5y - 13 = 0$.

8) Find the distance between parallel lines $9x + 6y - 7 = 0$. and $9x + 6y - 32 = 0$.

9) Find the equation of the line passing through the point of intersection of lines $x + y - 2 = 0$ and $2x - 3y + 4 = 0$ and making intercept 3 on the X–axis.

10) D(–1,8), E(4,–2), F(–5,–3) are midpoints of sides BC, CA and AB of ΔABC. Find

(i) equations of sides of ΔABC.

(ii) co–ordinates of the circumcenter of ΔABC.

MISCELLANEOUS EXERCISE - 5

1. Find the slopes of the lines passing through the following of points :

   (a) (1, 2), (3, –5) (b) (1, 3), (5, 2)

   (c) (–1, 3), (3, –1) (d) (2, –5), (3, –1)

2. Find the slope of the line which

   a) makes an angle of 30° with the positive X–axis.

   b) makes intercepts 3 and –4 on the axes.

   c) passes through the points $A(-2,1)$ and the origin .

3. Find the value of k

   a) if the slope of the line passing through the points (3, 4), (5, k) is 9.

   b) the points (1, 3), (4, 1), (3, k) are collinear

   c) the point P(1,k) lies on the line passing through the points A(2, 2) and B(3, 3).

4. Reduce the equation $6x+3y+8=0$ into slope–intercept form. Hence find its slope.

5. Verify that A(2, 7) is not a point on the line $x+2y+2=0$.

6. Find the X–intercept of the line $x+2y-1=0$.

7. Find the slope of the line $y-x+3=0$.

8. Does point $A(2,3)$ lie on the line $3x+2y-6=0$? Give reason.

9. Which of the following lines pass through the origin ?

   (a) $x=2$ (b) $y=3$

   (c) $y=x+2$ (d) $2x-y=0$

10. Obtain the equation of the line which is :

    a) parallel to the X–axis and 3 unit below it.

    b) parallel to the Y–axis and 2 unit to the left of it.

    c) parallel to the X–axis and making an intercept of 5 on the Y–axis.

    d) parallel to the Y–axis and making an intercept of 3 on the X–axis.

11. Obtain the equation of the line containing the point

    a) (2,3) and parallel to the X–axis.

    b) (2,4) and perpendicular to the Y–axis.

    c) (2,5) and perpendicular to the X–axis.

12. Find the equation of the line :

    a) having slope 5 and containing point $A(-1,2)$.

    b) containing the point (2, 1) and having slope 13.

    c) containing the point T(7,3) and having inclination 90^0.

    d) containing the origin and having inclination 90^0.

    e) through the origin which bisects the portion of the line $3x+2y=2$ intercepted between the co–ordinate axes.

13. Find the equation of the line passing through the points A(–3,0) and B(0,4).

14. Find the equation of the line :

a) having slope 5 and making intercept 5 on the X–axis.

b) having an inclination 60^0 and making intercept 4 on the Y–axis.

15. The vertices of a triangle are A(1,4), B(2,3) and C(1,6). Find equations of

(a) the sides (b) the medians

(c) Perpendicular bisectors of sides

(d) altitudes of ΔABC.

ACTIVITIES

Activity 5.1 :

Complete the activity and decide whether the line $2x - 4y = 5$ and $2x + y = 17$ are perpendicular to each other.

Slope of line $2x - 4y = 5$ is ☐.

Slope of line $2x + y = 17$ is ☐.

Product of slopes of these line = ☐ .

∴ The given lines are ☐ .

Activity 5.2 :

Complete the activity and obtain the equation of the line having slope 2 and which makes intercept 3 on the Y-axis.

Here c = ☐, m = ☐

The equation of line y = ☐ + c

∴ The equation of line is ☐

Activity 5.3 :

Complete the activity and find the equation of the line which is parallel to the line $3x + y = 5$ and passes through the origin.

Slope of line $3x + y = 5$ is ☐

The required line passes through the origin

It's equation is y = ☐

The equation of required line is ☐ .

Activity 5.4 :

Complete the following activity and find the value of p, if the area of the triangle formed by the axes and the line $3x + 4y = P$, is 24.

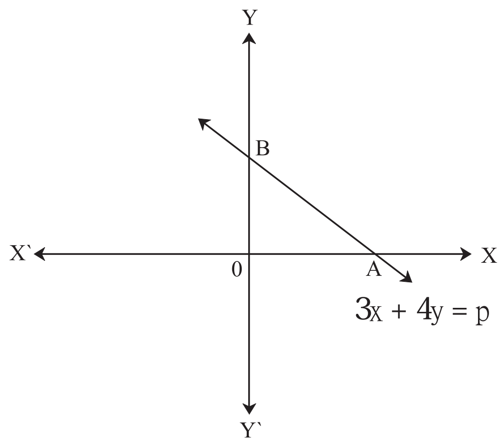


Fig. 5.15

Here, A = (☐, 0) B = (0, ☐)

$A(\Delta AOB) = \frac{1}{2}$ (OA) (OB)

∴ 24 = $\frac{1}{2}$ ☐ ☐

∴ P^2 = ☐

∴ P = ☐

❖ ❖ ❖

6. DETERMINANTS

Let's study.

- Definition of Determinants
- Properties of Determinants
- Applications of Determinants
- Cramer's Rule
- Consistency of linear equations
- Area of triangle
- Collinearity of three points

Let's recall.

In standard X, we have learnt to solve simultaneous equations in two variables using determinants of order two. We will now learn more about the determinants. It is useful in Engineering applications, Economics, etc.

> The concept of a determinant was discussed by the German Mathematician G.W. Leibnitz.
>
> Cramer developed the rule for solving linear equations using determinants.

The representation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as the determinant of order two. Numbers a, b, c, d are called elements of the determinant. In this arrangement, there are two rows and two columns.

$(a \quad b)$ is the 1st Row $\qquad$ $(c \quad d)$ is the 2nd Row

$\begin{pmatrix} a \\ c \end{pmatrix}$ is the 1st Column $\qquad$ $\begin{pmatrix} b \\ d \end{pmatrix}$ is the 2nd column

That is, four elements are enclosed between two vertical bars arranged in two rows and two columns.

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is associated with the expression $ad - bc$,

$ad-bc$ is called the value of the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Note that $ad-bc$ is an algebraic expression in a,b,c and d.

Let's learn.

6.1 Determinant of order 3 :

Definition : The determinant of order 3 is a square arrangement of 9 elements enclosed between two vertical bars. The elements are arranged in 3 rows and 3 columns as given below.

$$\begin{array}{c} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \\ \begin{matrix} C_1 & \quad C_2 & \quad C_3 \end{matrix} \end{array}$$

Here a_{11}, a_{12}, a_{13} are element in row I (R_1)

a_{21}, a_{22}, a_{23} are elements in row II (R_2)

a_{31}, $a_{32,}$ a_{33} are elements in row III (R_3)

Similarly a_{11}, a_{21}, a_{31} are elements in column I (C_1) and so on.

Here $\quad a_{ij}$ represents the element in i^{th} row and j^{th} column of the determinant.

For example : a_{31} represents the element in 3rd row and 1st column.

In general, we denote determinant by D or A or Δ (delta) .

For example, $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Like a 2×2 determinant, 3×3 determinant is also an algebraic expression in terms of elements of the determinant. We find that expression by expanding the determinant.

6.1.1 Expansion of Determinant

There are six ways of expanding a determinant of order 3, corresponding to each of three rows (R_1, R_2 , R_3) and three columns (C_1, C_2, C_3). We give here the expansion by the 1st row of the determinant D

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The determinant can be expanded as follows:

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For example,

$$D = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

We expand the determinant as follows in two different ways.

$$(1)\ (D) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2-1) + 2(-1-0) + 4(1-0)$$

$$= 3(-3) + 2(-1) + 4(1)$$

$$= -9 - 2 + 4 = -11 \ \ +4 = -7$$

$$(2)\ D = (a_{11}\cdot a_{22}\cdot a_{33}) + (a_{12}\cdot a_{23}\cdot a_{31}) + (a_{13}\cdot a_{32}\cdot a_{21}) - (a_{13}\cdot a_{22}\cdot a_{31}) - (a_{12}\cdot a_{21}\cdot a_{33}) - (a_{11}\cdot a_{32}\cdot a_{23})$$

$$= (3\times2\times-1) + (-2\times1\times0) + (4\times1\times1) - (4\times2\times0) - (-2\times1\times-1) - (3\times1\times1)$$

$$= -6 + 0 + 4 - 0 - 2 - 3 = -11 + 4 = -7$$

SOLVED EXAMPLES

Ex.1 Evaluate the following determinants :

1) $\begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$ 2) $\begin{vmatrix} -2 & 12 \\ 3 & 25 \end{vmatrix}$

3) $\begin{vmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{vmatrix}$ 4) $\begin{vmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix}$

Solution :

1) $\begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 15 - 14 = 1$

2) $\begin{vmatrix} -2 & 12 \\ 3 & 25 \end{vmatrix} = -50 - 36 = -86$

3) $\begin{vmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{vmatrix}$

$$= 3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 3(-1) - 2(-1) + 6(0) = -1$$

4) $\begin{vmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 3 & 3 \end{vmatrix}$

$$= 1(-8) - 3(-6) + 3(6)$$

$= -8+18+18=28$

Ext. 2. Find the value of x if

1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 4 & 6 & 9 \end{vmatrix} = 0$ 2) $\begin{vmatrix} 1 & 1 & -x \\ x & -4 & 5 \\ x & -2 & 1 \end{vmatrix} = 0$

Solution :

1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 4 & 6 & 9 \end{vmatrix} = 0$

$\therefore$ $1(18-24) - x(9-16) + x^2(6-8) = 0$

$\therefore$ $1(-6) - x(-7) + x^2(-2) = 0$

$\therefore$ $-6 + 7x - 2x^2 = 0$

$\therefore$ $2x^2 - 7x + 6 = 0$

$\therefore$ $(2x-3)(x-2) = 0$

$\therefore$ $x = \frac{3}{2}$ or $x = 2$

2) $\begin{vmatrix} 1 & 1 & -x \\ x & -4 & 5 \\ x & -2 & 1 \end{vmatrix} = 0$

$\therefore$ $1(-4+10) - 1(x-5x) - x(-2x+4x) = 0$

$\therefore$ $1(6) - 1(-4x) - x(2x) = 0$

$\therefore$ $6 + 4x - 2x^2 = 0$

$\therefore$ $2x^2 - 4x - 6 = 0$

$\therefore$ $(2x+2)(x-3) = 0$

$\therefore$ $x = -1$ or $x = 3$

EXERCISE 6.1

1) Evaluate the following determinants :

i) $\begin{vmatrix} 4 & 7 \\ -7 & 0 \end{vmatrix}$ ii) $\begin{vmatrix} 3 & -5 & 2 \\ 1 & 8 & 9 \\ 3 & 7 & 0 \end{vmatrix}$ iii) $\begin{vmatrix} 1 & i & 3 \\ i^3 & 2 & 5 \\ 3 & 2 & i^4 \end{vmatrix}$

iv) $\begin{vmatrix} 5 & 5 & 5 \\ 5 & 4 & 4 \\ 5 & 4 & 8 \end{vmatrix}$ v) $\begin{vmatrix} 2i & 3 \\ 4 & -i \end{vmatrix}$ vi) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

vii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ viii) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

2) Find the value of x if

i) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

ii) $\begin{vmatrix} 2 & 1 & x+1 \\ -1 & 3 & -4 \\ 0 & -5 & 3 \end{vmatrix} = 0$ iii) $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

3) Solve the following equations.

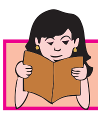
i) $\begin{vmatrix} x & 2 & 2 \\ 2 & x & 2 \\ 2 & 2 & x \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

4) Find the value of x if

$\begin{vmatrix} x & -1 & 2 \\ 2x & 1 & -3 \\ 3 & -4 & 5 \end{vmatrix} = 29$

5) Find x and y if $\begin{vmatrix} 4i & i^3 & 2i \\ 1 & 3i^2 & 4 \\ 5 & -3 & i \end{vmatrix} = x + iy$

where $i = \sqrt{-1}$



Let's learn.

6.2 PROPERTIES OF DETERMINANTS

In the previous section we have learnt how to expand a determinant. Now we will study some properties of determinants. They will help us to evaluate a determinant more easily.

Property 1 - The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

Let D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 .(b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1 a_3 b_2$ ———————— (i)

Let D$_1$ $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - c_1 b_3) + a_3(b_1 c_2 - c_1 b_2)$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 c_1 b_3 + a_3 b_1 c_2 - a_3 c_1 b_2$

———————— (ii)

From (i) and (ii) D= D$_1$

For Example,

Let A $= \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$

$= 1 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix}$

$= 1(-1-4) - 2\,(3-0) - 1(6-0)$

$= -5 - 6 - 6$

$= -17$ ………………………….. (i)

By interchanging rows and columns of A we get the determinant A$_1$

A$_1$ $= \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ -1 & 2 & 1 \end{vmatrix}$

$= 1 \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= 1(-1-4) - 3\,(2+2) + 0 = -5 - 12 = -17$

………………………………...(ii)

$\therefore$ A = A$_1$ from (i) and (ii)

Property 2 - If any two rows (or columns) of a determinant are interchanged then the value of the determinant changes only in sign.

Let D $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 .(b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1 a_3 b_2$

Let D$_1$ = determinant obtained by interchanging first and second row of determinant D

D$_1$ $= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_2 (b_1 c_3 - b_3 c_1) - b_2 (a_1 c_3 - a_3 c_1) + c_2(a_1 b_3 - a_3 b_1)$

$= a_2 b_1 c_3 - a_2 b_3 c_1 - b_2 a_1 c_3 + b_2 a_3 c_1 + c_2 a_1 b_3 - c_2 a_3 b_1$

$= -[a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1 a_3 b_2]$

$= -$ D

For Example,

Let A $= \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$

$= 1(-1-4) - 2\,(3-0) - 1(6-0)$

$= -5-6-6$

$= -17$

Interchange 1^{st} and 3^{rd} rows. Then new determinant is given by A_1

$$A_1 = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 0(1-4) - 2(-3-2)+1(6+1)$

$= 0+10+7$

$= 17$

$\therefore A_1 = -A$

Interchange 2^{nd} and 3^{rd} column in A and let the new determinant obtained be A_2

$$\therefore A_2 = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 1(4+1) +1(6)+2(3)$

$= 5+6+6 = 17 \quad \therefore A_2 = -A$

Note – We denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$

Property 3 - If any two rows (or columns) of a determinant are identical then the value of the determinant is zero

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Here first and second rows are identical.

$D = a_1 .(b_1 c_3 - b_3 c_1) - b_1 (a_1c_3 - a_3c_1) + c_1(a_1 b_3 - a_3b_1)$

$= a_1b_1 c_3 - a_1b_3 c_1 - b_1a_1c_3 + b_1 a_3c_1 + c_1a_1 b_3 - c_1 a_3b_1$

$= 0$

$$\text{For Example, } A = \begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix}$$

$= 1(-2-2) -(-1) (2-0) + 2(1-0)$

$= -4 + 2 + 2 = -4 + 4 = 0$

Here in determinant A first and second rows are identical

Property 4 - If each element of a row (or column) of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$ .... (i)

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$= a_1 (kb_2c_3 - kb_3c_2) - b_1 (ka_2c_3 - ka_3c_2) + c_1 (ka_2b_3 - ka_3b_2)$

$= k[a_1 (b_2c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2b_3 - a_3b_2)]$

$= k D$ (from i)

$$\text{For Example, } A = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 0 - 2(-3-2) + 1(6+1)$

$= +10 + 7$

$= 17$

Multiply R_2 by 3 we get,

$$A_1 = \begin{vmatrix} 0 & 2 & 1 \\ 3\times3 & -1\times3 & 2\times3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 & 1 \\ 9 & -3 & 6 \\ 1 & 2 & -1 \end{vmatrix} = 0 - 2(-9-6) + 1(18+3)$$

$= 30 + 21 = 51$

$= 3 \times 17$

$\therefore A_1 = 3 A$

Remark i) Using this property we can take out a common factor from any one row (or any one column) of the given determinant.

ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio) then the value of the determinant is zero.

Example :

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 8 & 16 & 24 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 8\times1 & 8\times2 & 8\times3 \end{vmatrix} = 8 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 1 & 2 & 3 \end{vmatrix}$$

(using property 4)

$= 8 \times 0$ (using property 3)

Activity 6.1

$$\text{Simplify : } A = \begin{vmatrix} 25 & 75 & 125 \\ 13 & 26 & 39 \\ 17 & 51 & 34 \end{vmatrix}$$

$$A = \square \times \square \times \square \begin{vmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \square \,[1 \times (\square - \square) - 1(6-15)$$

$$+1\,(\square - \square)]$$

$$= \square \times \square = \square$$

Property 5 - If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants

Example (i)

$$\begin{vmatrix} a_1+x_1 & b_1+y_1 & c_1+z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example (ii)

$$\begin{vmatrix} x_1+y_1+z_1 & p & l \\ x_2+y_2+z_2 & q & m \\ x_3+y_3+z_3 & r & n \end{vmatrix} = \begin{vmatrix} x_1 & p & l \\ x_2 & q & m \\ x_3 & r & n \end{vmatrix} + \begin{vmatrix} y_1 & p & l \\ y_2 & q & m \\ y_3 & r & n \end{vmatrix} + \begin{vmatrix} z_1 & p & l \\ z_2 & q & m \\ z_3 & r & n \end{vmatrix}$$

Property 6 - If a constant multiple of all elements of any row (or column) is added to the corresponding elements of any other row (or column) then the value of new determinant so obtained is the same as that of the original determinant.

Note - In all the above properties we perform some operations on rows (or columns). We indicate these operations in symbolic form as below.

i) $R_i \leftrightarrow R_j$ means interchange i^{th} and J^{th} rows.

ii) $C_i \leftrightarrow C_j$ means interchange i^{th} and j^{th} columns.

iii) $R_i \rightarrow kR_i$ (or $C_i \rightarrow kC_i$) means multiplying i^{th} row (or i^{th} column) by constant k.

iv $R_i \rightarrow R_j + kR_J$ (or $C_i \rightarrow C_i + kC_j$) means change in i^{th} row (or i^{th} column) by adding k multiples of corresponding elements of j^{th} row (or j^{th} column) in i^{th} row (or i^{th} column).

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Performing $R_1 \rightarrow R_1 + k.R_3$

$$A_1 = \begin{vmatrix} a_1+ka_3 & b_1+kb_3 & c_1+kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(by property 5)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(by property 4)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0)$$

(R_1 and R_3 are identical)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = A$$

Example : Let $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(2-0) - 2(-1-0) + 3(-2-2)$

$= 2 + 2 - 12$

$= 4 - 12 = -8$ ——————————(i)

Now, $B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$

$R_1 \to R_1 + 2R_2$. Then

$$B_1 = \begin{vmatrix} 1+2(-1) & 2+2(2) & 3+2(0) \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$B_1 = \begin{vmatrix} -1 & 6 & 3 \\ -1 & 2 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -1(2-0) - 6(-1-0) + 3(-2-2)$$

$= -2 + 6 - 12 = 6 - 14 = -8$ ——–—(ii)

From (i) and (ii) $B = B_1$

Remark – If more than one operations are to be done, make sure that the operations are done one at a time. Else there may be a mistake in calculations.

The diagonal of a determinant : The diagonal (main or principal diagonal) of a determinant A is collection of entries a_{ij} where $i = j$

OR

The set of elements (a_{11} , a_{22}, a_{33} , ... a_{nn}) forms the diagonal of a determinant A

e.g. $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Here a_{11}, a_{22}, a_{33} are element of diagonal.

Property 7 - (Triangle property) – If all the elements of a determinant above or below the diagonal are zero then the value of the determinant is equal to the product of its diagonal elements.

Verification:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

For example, $A = \begin{vmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 2 \end{vmatrix} = a_1 b_2 c_3 = 3 \times 4 \times 2 = 24$

SOLVED EXAMPLES

Ext. 1. Without expanding determinant find the value of :

i) $\begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$ ii) $\begin{vmatrix} 17 & 18 & 19 \\ 20 & 21 & 22 \\ 23 & 24 & 25 \end{vmatrix}$

Solution :

i) Let $D = \begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$

Performing $R_1 \to R_1 + 4R_2$

$$= \begin{vmatrix} 11 & 0 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$$

Taking 11 common from R_1.

$$= 11\begin{vmatrix}1 & 0 & -1\\ 2 & -1 & 0\\ 5 & -2 & -1\end{vmatrix} \quad R_3 \to R_3 - 2R_2$$

$$= 11\begin{vmatrix}1 & 0 & -1\\ 2 & -1 & 0\\ 1 & 0 & -1\end{vmatrix}$$

$= 11\ (0) = 0....$

$\because$ R_1 and R_3 are indentical.

ii) Let D $= \begin{vmatrix}17 & 18 & 19\\ 20 & 21 & 22\\ 23 & 24 & 25\end{vmatrix}$

Performing $C_2 \to C_2 - C_1$

$$= \begin{vmatrix}17 & 1 & 19\\ 20 & 1 & 22\\ 23 & 1 & 25\end{vmatrix}$$

Performing $C_3 \to C_3 - 2C_2$

$$= \begin{vmatrix}17 & 1 & 17\\ 20 & 1 & 20\\ 23 & 1 & 23\end{vmatrix}$$

$=$ 0, since C_1 and C_3 are identical.

Ex 2. By using properties of determinant show that

i) $\begin{vmatrix}p & q & r\\ a & b & c\\ x & y & z\end{vmatrix} = \begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix}$

ii) $\begin{vmatrix}x+y & y+z & z+x\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix} = 2\begin{vmatrix}x & y & z\\ z & x & y\\ y & z & x\end{vmatrix}$

Solution :

i) LHS $= \begin{vmatrix}p & q & r\\ a & b & c\\ x & y & z\end{vmatrix}$ Performing $C_1 \leftrightarrow C_3$

$$= (-1)\begin{vmatrix}r & p & q\\ c & b & a\\ z & y & x\end{vmatrix} \quad \text{Performing } C_2 \leftrightarrow C_3$$

$$= (-1)(-1)\begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix}$$

$$= \begin{vmatrix}r & p & q\\ c & a & b\\ z & x & y\end{vmatrix} = \text{RHS}$$

ii) RHS $= \begin{vmatrix}x+y & y+z & z+x\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix}2(x+y+z) & 2(x+y+z) & 2(x+y+z)\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$= 2\begin{vmatrix}x & y & z\\ z+x & x+y & y+z\\ y+z & z+x & x+y\end{vmatrix} \quad R_2 \to R_2 - R_1$$

$$= 2\begin{vmatrix}x & y & z\\ z & x & y\\ y+z & z+x & x+y\end{vmatrix} \quad R_3 \to R_3 - R_2$$

$$= 2\begin{vmatrix}x & y & z\\ z & x & y\\ y & z & x\end{vmatrix} = \text{RHS}$$

Ex 3. Solve the equation $\begin{vmatrix} x+1 & x+2 & 3 \\ 3 & x+2 & x+1 \\ x+1 & 2 & x+3 \end{vmatrix} = 0$

Solution :

$$\begin{vmatrix} x+1 & x+2 & 3 \\ 3 & x+2 & x+1 \\ x+1 & 2 & x+3 \end{vmatrix} = 0 \quad R_2 \to R_2 - R_1$$

$$\therefore \begin{vmatrix} x+1 & x+2 & 3 \\ 2-x & 0 & x-2 \\ x+1 & 2 & x+3 \end{vmatrix} = 0 \quad R_3 \to R_3 - R_1$$

$$\therefore \begin{vmatrix} x+1 & x+2 & 3 \\ 2-x & 0 & x-2 \\ 0 & -x & x \end{vmatrix} = 0 \quad C_2 \to C_2 + C_3$$

$$\therefore \begin{vmatrix} x+1 & x+5 & 3 \\ 2-x & x-2 & x-2 \\ 0 & 0 & x \end{vmatrix} = 0$$

$$\therefore x(x-2)\begin{vmatrix} x+1 & x+5 & 3 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$\therefore x(x-2)[(x+1)(1) - (x+5)(-1) + 3(0)] = 0$

$\therefore x(x-2)(2x+6) = 0$

$\therefore 2x(x-2)(x+3) = 0$

$\therefore x = 0$ or $x = 2$ or $x = -3$

EXERCISE 6.2

1) Without expanding evaluate the following determinants.

i) $\begin{vmatrix} 1 & a & a+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ iii) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

2) Using properties of determinant show that

$$\begin{vmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{vmatrix} = 4abc$$

3) Solve the following equation.

$$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

4) If $\begin{vmatrix} 4+x & 4-x & 4-x \\ 4-x & 4+x & 4-x \\ 4-x & 4-x & 4+x \end{vmatrix} = 0$ then find the values of x

5) Without expanding determinants show that

$$\begin{vmatrix} 1 & 3 & 6 \\ 6 & 1 & 4 \\ 3 & 7 & 12 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 7 & 6 \end{vmatrix} = 10\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 7 \\ 3 & 2 & 6 \end{vmatrix}$$

6) Without expanding determinants find the value of

i) $\begin{vmatrix} 10 & 57 & 107 \\ 12 & 64 & 124 \\ 15 & 78 & 153 \end{vmatrix}$ ii) $\begin{vmatrix} 2014 & 2017 & 1 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 1 \end{vmatrix}$

7) Without expanding determinant prove that

i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

ii) $\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

6.4 APPLICATIONS OF DETERMINANTS

6.4.1 Cramer's Rule

In linear algebra Cramer's rule is an explicit formula for the solution of a system of linear equations in many variables. In previous class we have studied this with two variables. Our goal here is to extend the application of Cramer's rule to three equations in three variables (unknowns) . Variables are usually denoted by x, y and z.

Theorem – If $a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

are three linear equations in three variables x, y and z; a_i , b_i , c_i are coefficients of x,y,z and di are constants. Then solutions of the equations are

$$x = \frac{Dx}{D},\ y = \frac{Dy}{D},\ z = \frac{Dz}{D}$$

provided D ≠ 0 and where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad Dy = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$Dx = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad Dz = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Remark : If D = 0 then there is no unique solution for the given system of equations.

SOLVED EXAMPLES

1) Solve the following equations using Cramer's Rule

1) $2x - y + z = 1,\quad x + 2y + 3z = 8,\quad 3x + y - 4z = 1$

2) $x + y - z = 2,\quad x - 2y + z = 3,\quad 2x - y - 3z = -1$

Solution :

1) Given system is

$2x - y + z = 1$

$x + 2y + 3z = 8$

$3x + y - 4z = 1$

$$\therefore D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$= 2(-8-3) + 1(-4-9) + 1(1-6)$

$= 2(-11) + 1(-13) + 1(-5) = -40$

$$\therefore D_x = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 2 & 3 \\ 1 & 1 & -4 \end{vmatrix}$$

$= 1(-8-3) + 1(-32-3) + 1(8-2)$

$= -11 - 35 + 6 = -40$

$$\therefore D_y = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$= 2(-32-3) - 1(-4-9) + 1(1-24)$

$= -70 + 13 - 23 = -80$

$$\therefore D_z = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 8 \\ 3 & 1 & 1 \end{vmatrix}$$

$= 2(2-8) + 1(1-24) + 1(1-6)$

$= -12 - 23 - 5 = -40$

$$x = \frac{D_x}{D} = \frac{-40}{-40} = 1 \quad y = \frac{D_y}{D} = \frac{-80}{-40} = 2$$

$$z = \frac{D_z}{D} = \frac{-40}{-40} = 1,\ \therefore x = 1,\ y = 2,\ z = 1$$

2) Given system is

$x + y - z = 2;\quad x - 2y + z = 3;$

$2x - y - 3z = -1$

$$\therefore D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$= 1(6+1) - 1(-3-2) - 1(-1+4)$

$= 7 + 5 - 3 = 9$

$$D_x = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ -1 & -1 & -3 \end{vmatrix}$$

$= 2(7) - 1(-8) - 1(-5) = 14 + 8 + 5 = 27$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$= 1(-8) - 2(-5) - 1(-7) = -8 + 10 + 7 = 9$

$$\therefore D_z = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix}$$

$= 1(5) - 1(-7) + 2(3) = 5 + 7 + 6 = 18$

$$x = \frac{D_x}{D} = \frac{27}{9} = 3 \quad y = \frac{D_y}{D} = \frac{9}{9} = 1$$

$$z = \frac{D_z}{D} = \frac{18}{9} = 2$$

$\therefore \quad x = 3, y = 1, z = 2$

6.4.2 Consistency of three linear equations in two variables

Consider the system of three linear equations in two variables x and y

$$\left.\begin{aligned} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \\ a_3x + b_3y + c_3 = 0 \end{aligned}\right\} \ldots\ldots (I)$$

These three equations are said to be consistent if they have a common solution.

The ***necessary*** condition for system (I) to be consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

SOLVED EXAMPLES

Ex. 1) Show that the equations are consistent $3x + 4y = 11$, $2x - y = 0$ and $5x - 2y = 1$

Solution : By condition of consistency consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -11 \\ 2 & -1 & 0 \\ 5 & -2 & -1 \end{vmatrix}$$

$= 3(1 - 0) - 4(-2 - 0) - 11(-4 + 5) = 3 + 8 - 11$

$= 11 - 11 = 0$

$\therefore$ the given system of equations is consistent

Ex. 2) Show that the following equations are not consistent

$x+2y = 1$, $x-y = 2$ and $x-2y = 0$

Solution : For consistency of equations, consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -2 & 0 \end{vmatrix}$$

$= 1(0+4) - 2(0-2) - 1(-2+1) = 4+4+1 = 9 \neq 0$

$\therefore$ the given system of equations is not consistent.

Note : Consider the following equations

$2x + 2y = -3, \quad x + y = -1, \quad 3x + 3y = -5$

The given equations in the standard form are

$2x + 2y + 3 = 0, \quad x + y + 1 = 0, \quad 3x + 3y + 5 = 0$

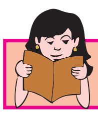
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$$

$= 0 \qquad (\text{As } R_1 = R_2)$

The slopes of the lines $2x + 2y+2 = 0$ and $x + y + 1 = 0$ are equal and each is (-1)

So, the slope of each of these lines is - 1. Threrefore, the lines are parallel. So they do not have a common solution and hence the system of equations is not consistent.

Thus D = 0 is not a sufficient condition for the consistency of three equations in two variables.



Let's learn.

6.4.3 Area of a triangle and Collinearity of three points.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of triangle ABC then the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

Remark:

i) Area is a positive quantity. Hence we always take the absolute value of the determinant.

ii) If area is given, consider both positive and negative values of determinant for calculation of unknown co–ordinates.

iii) If the value of the determinant is zero then the area of Δ ABC is zero. It implies that the points A, B and C are **collinear**.

SOLVED EXAMPLES

Q.1) Find the area of the triangle whose vertices are $P(2, -6)$, $Q(5, 4)$, $R(-2, 4)$

Solution: Let, $P(2,-6) \equiv (x_1,y_1)$, $Q(5,4) \equiv (x_2,y_2)$, $R(-2,4) \equiv (x_3,y_3)$.

Area of the triangle PQR

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [2(0) + 6(7) + 1(28)]$$

$$= \frac{1}{2} [70]=35 \quad \text{square unit.}$$

Q. 2) Using determinant show that following points are collinear.

$A(2,3)$, $B(-1,-2)$, $C(5,8)$

Solution:

$$A(\Delta ABC) = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [2(-10) -3(-6) + 1(2)]$$

$$= \frac{1}{2} [-20 + 18 + 2] = 0$$

∴ Points A, B, C are collinear.

Q.3) If $P(3,-5)$, $Q(-2,k)$, $R(1,4)$ are vertices of triangle PQR and its area is $\frac{17}{2}$ square unit then find the value of k.

Solution : Let $P(3,-5) \equiv (x_1,y_1)$, $Q(-2,k) \equiv (x_2,y_2)$, $R(1,4) \equiv (x_3,y_3)$.

$$A(\Delta PQR) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & -5 & 1 \\ -2 & k & 1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [3(k - 4) + 5(-3) + 1(-8 - k)]$$

$$= \frac{1}{2} [3k - 12 - 15 - 8 - k] = \frac{1}{2} [2k - 35]$$

But area of triangle PQR is $\frac{17}{2}$

$$\therefore \frac{1}{2} [2k - 35] = \pm \frac{17}{2}$$

$\therefore 2k - 35 = \pm 17$

$\therefore 2k = 52$ or $2k = 18$

$\therefore k = 26$ or $k = 9$

Q.4) Find the area of the triangle whose vertices are A(–2, –3) , B(3, 2) and C(–1, –8)

solution : Given $(x_1, y_1) \equiv (-2, -3)$, $(x_2, y_2) \equiv (3, 2)$, and $(x_3, y_3) \equiv (-1 , -8)$

We know that area of a triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [-2(2 + 8) + 3(3 + 1) + 1(-24 + 2)]$$

$$= \frac{1}{2} [-20 + 12 - 22] = \frac{1}{2} [-42 + 12]$$

$= \frac{1}{2}[-30] \quad = -15$

But Area is positive.

$\therefore$ Area of triangle = 15 square unit

Q.5) If the area of a triangle with vertices $P(-3, 0)$, $Q(3, 0)$ and $R(0, k)$ is 9 square unit then find the value of k.

Solution : Given $(x_1, y_1) \equiv (-3, 0)$, $(x_2, y_2) \equiv (3, 0)$ and $(x_3, y_3) \equiv (0, k)$ and the area of the triangle is 9 sq. unit.

We know that area of a triangle

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 9$$

(Area is positive but the value of the determinant can be positive or negative)

$$\therefore \frac{1}{2}[-3(0-k) + 1(3k-0)] = \pm 9$$

$$\therefore \frac{1}{2}[6k] = \pm 9,$$

$$\therefore 6k = \pm 18,$$

$$\therefore k = \pm 3$$

EXERCISE 6.3

1) Solve the following equations using Cramer's Rule.

i) $x + 2y - z = 5, \quad 2x - y + z = 1,$
$3x + 3y = 8$

ii) $2x - y + 6z = 10, 3x + 4y - 5z = 11,$
$8x - 7y - 9z = 12$

iii) $11x - y - z = 31, x - 6y + 2z = -26,$
$x + 2y - 7z = -24$

iv) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -2$

$\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 3$

$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = -1$

v) $\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = 4, \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 2, \frac{3}{x} + \frac{1}{y} - \frac{1}{z} = 2$

2) An amount of Rs. 5000 is invested in three plans at rates 6%, 7% and 8% per annuam respectively. The total annual income from these investments is Rs 350. If the total annual income from first two investments is Rs. 70 more than the income from the third, find the amount invested in each plan by using Cramer's Rule.

3) Show that following equations are consistent.
$2x + 3y + 4 = 0, x + 2y + 3 = 0,$
$3x + 4y + 5 = 0$

4) Find k if the following equations are consistent.

i) $x + 3y + 2 = 0, 2x + 4y - k = 0,$
$x - 2y - 3k = 0$

ii) $(k-1)x + (k-1)y = 17,$
$(k-1)x + (k-2)y = 18, x + y = 5$

5) Find the area of the triangle whose vertices are :

i) (4,5), (0,7), (–1,1)

ii) (3,2,), (–1,5), (–2,–3)

iii) (0,5), (0,–5), (5,0)

6) Find the value of k if the area of the triangle with vertices $A(k,3)$, $B(-5,7)$, $C(-1,4)$ is 4 square unit.

7) Find the area of the quadrilateral whose vertices are $A(-3,1)$, $B(-2,-2)$, $C(4,1)$, $D(2,3)$.

8) By using determinant show that following points are collinear.
$P(5,0)$, $Q(10,-3)$, $R(-5,6)$

9) The sum of three numbers is 15. If the second number is subtracted from the sum of first and third numbers then we get 5. When the third number is subtracted from the sum of twice the first number and the second number, we get 4. Find the three numbers.

Let's remember!

1) The value of determinant of order 3×3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

2) **Properties of determinant**

(i) The value of determinant remains unchanged if its rows are turned into columns and columns are turned into rows.

(ii) If any two rows (or columns) of a determinant are interchanged then the value of the determinant changes its sign.

(iii) If any two rows (or columns) of a determinant are identical then the value of the determinant is zero

(iv) If each element of a row (or column) of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the old determinant

(v) If each element of a row (or column) is expressed as the sum of two numbers then the determinant can be expressed as a sum of two determinants

(vi) If a constant multiple of all elements of a row (or column) is added to the corresponding elements of any other row (or column) then the value of new determinant so obtained is the same as that of the original determinant.

(vii) (Triangle property) – If all element of the determinant above or below the main diagonal are zero then the value of the determinant is equal to product of its diagonal elements.

3) Solve the system of linear equation using Cramer's Rule.

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D} \quad \text{provided } D \neq 0$$

4) Consistency of three equations.

If $a_1x + b_1y + c_1 = 0 \quad a_2x + b_2y + c_2 = 0$

$a_3x + b_3y + c_3 = 0$ are consistent, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

5) Area of a triangle, the vertices of which are $(x_1, y_1), (x_2,y_2)\ (x_3,y_3)$; is

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

6) Points $(x_1, y_1), (x_2,y_2)\ (x_3,y_3)$ are collinear if and only if

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

MISCELLANEOUS EXERCISE - 6

Q.1) Evaluate i) $\begin{vmatrix} 2 & -5 & 7 \\ 5 & 2 & 1 \\ 9 & 0 & 2 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & -3 & 12 \\ 0 & 2 & -4 \\ 9 & 7 & 2 \end{vmatrix}$

Q.2) Find the value of x if

i) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & -5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ ii) $\begin{vmatrix} 1 & 2x & 4x \\ 1 & 4 & 16 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q.3) By using properties of determinant prove

that $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q.4) Without expanding determinant as far as possible, show that

i) $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$

ii) $\begin{vmatrix} xa & yb & zc \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ a & b & c \\ bc & ca & ab \end{vmatrix}$

iii) $\begin{vmatrix} l & m & n \\ e & d & f \\ u & v & w \end{vmatrix} = \begin{vmatrix} n & f & w \\ l & e & u \\ m & d & v \end{vmatrix}$

iv) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$

Q.5) Solve the following linear equations by Cramer's Rule.

i) $2x-y+z = 1$, $x+2y+3z = 8$, $3x+y-4z = 1$

ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -2$

$\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 3$

$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = -1$

iii) $x-y+2z=7$, $3x+4y-5z=5$, $2x-y+3z=12$

Q.6) Find the value of k if the following equation are consistent.

i) $3x+y-2=0$, $kx+2y-3=0$ and $2x-y=3$

ii) $kx + 3y + 4 = 0$

$x + ky + 3 = 0$

$3x + 4y + 5 = 0$

Q.7) Find the area of triangle whose vertices are

i) A(–1,2) ,B(2,4) ,C(0,0)

ii) P(3,6) , Q(–1,3) , R (2,–1)

iii) L(1,1) , M (–2,2), N (5,4)

Q.8) Find the value of k.

i) if area of triangle Δ PQR is 4 square unit and vertices are $P(k, 0)$, $Q(4, 0)$, $R(0, 2)$

ii) if area of Δ LMN is 33/2 square unit and vertices are L (3,–5), $M(-2,k)$, N (1,4)

ACTIVITIES

Activity 6.1 :

Apply cramer's Rule for the following linear equation $x + y - z = 1$, $8x + 3y - 6z = 1$, $-4x - y + 3z = 1$.

Solution:

$D = \begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix} = \square$

$D_x = \begin{vmatrix} \square & 1 & -1 \\ \square & 3 & -6 \\ \square & -1 & 3 \end{vmatrix} = \square$

$D_y = \begin{vmatrix} 1 & \square & -1 \\ 8 & \square & -6 \\ -4 & \square & 3 \end{vmatrix} = \square$

$D_z = \begin{vmatrix} 1 & 1 & \square \\ 8 & 3 & \square \\ -4 & -1 & \square \end{vmatrix} = \square$

$$x = \frac{Dx}{D} = \frac{\Box}{\Box} = \Box$$

$$y = \frac{\Box}{D} = \frac{\Box}{\Box} = \Box$$

$$z = \frac{Dz}{D} = \frac{\Box}{\Box} = \Box$$

Activity 6.2 :

Fill in the blanks in the steps of solution of the following problem and complete it.

The cost of 4 kg potato, 3 kg wheat and 2 kg rice is Rs. 150. The cost of 1 kg potato, 2 kg wheat and 3 kg rice is Rs. 125. The cost of 6 kg potato, 2 kg wheat and 3 kg rice is Rs. 175. Find the cost of each item per kg by using Cramer's Rule.

Solution : Let x,y,z be the costs of potato, wheat and rice per kg respectively. The given information can be written in equation form as

$4x + 3y + 2z = \ldots$
$x + \ldots y + 3z = 125$
$\ldots x + 2y + 3z = 175$

$$D = \begin{vmatrix} 4 & 3 & 2 \\ 1 & \ldots & 3 \\ 6 & 2 & 3 \end{vmatrix} = 25$$

$$D_x = \begin{vmatrix} \ldots & 3 & 2 \\ 125 & 2 & 3 \\ 175 & 2 & 3 \end{vmatrix} = 250$$

$$D_y = \begin{vmatrix} 4 & 150 & 2 \\ 1 & 125 & 3 \\ 6 & 175 & 3 \end{vmatrix} = \ldots$$

$$D_z = \begin{vmatrix} 4 & 3 & 150 \\ 1 & \ldots & 125 \\ 6 & 2 & 175 \end{vmatrix} = 625$$

$$x = \frac{D_x}{D} = \ldots \quad y = \frac{D_y}{D} = \ldots \quad z = \frac{D_z}{D} = \ldots$$

Activity 6.3 :

Find the equation of the line joining the points $P(2,-3)$ & $Q(-4,1)$ using determinants.

Solution: $P(2,-3)$ & $Q(-4,1)$, $R(\Box, \Box)$

$\because$ Area of Δ PQR = 0

$$\Box \begin{vmatrix} \Box & \Box & 1 \\ \Box & \Box & 1 \\ \Box & \Box & 1 \end{vmatrix} = 0$$

$\Box$ = 0 is the eq^n of line

Activity 6.4 :

Find k, if the following equations are consistent.

$x + y = 0, \;\; kx - 4y + 5 = 0, \; kx - 2y + 1 = 0$

Solution:

$$\begin{vmatrix} \Box & \Box & \Box \\ \Box & \Box & 5 \\ \Box & \Box & 1 \end{vmatrix} = 0$$

$6 + \Box = 0$

$4k = \Box$

$k = -\Box$

❖ ❖ ❖

7. LIMITS

Let's study.

- Definition of Limit
- Algebra of Limits
- Evaluation of Limits
  - ➢ Direct Method
  - ➢ Factorization Method
  - ➢ Rationalization Method
- Limits of Exponential and Logarithmic Functions

Introduction:

Calculus is one of the important branches of Mathematics. The concept of limit of a function is a fundamental concept in calculus.

Let's understand

Meaning of $x \to a$:

When x takes the values gradually nearer to a, we say that 'x tends to a'. This is symbolically written as '$x \to a$'.

'$x \to a$' implies that $x \neq a$ and hence $(x-a) \neq 0$

Limit of a function :

Let us understand the concept by an example.

Consider the function $f(x) = x + 3$

Take the value of x very close to 3, but not equal to 3; and observe the values of $f(x)$.

	x approaches 3 from left				
x	2.5	2.6	…	2.9	2.99
$f(x)$	5.5	5.6	…	5.9	5.99

	x approaches 3 from right				
x	3.6	3.5	…	3.1	3.01
$f(x)$	6.6	6.5	…	6.1	6.01

From the table we observe that as $x \to 3$ from either side. $f(x) \to 6$.

This idea can be expressed by saying that the limiting value of $f(x)$ is 6 when x approches to 3.

This is symbolically written as,

$$\lim_{x \to 3} f(x) = 6$$

$$\text{i.e.} \quad \lim_{x \to 3} (x+3) = 6$$

Thus, limit of the function, $f(x) = x + 3$ as $x \to 3$ is the value of the function at $x = 3$.

Let's learn.

7.1 DEFINITION OF LIMIT OF A FUNCTION:

A function $f(x)$ is said to have the limit l as x tends to a, if for every $\in > 0$ we can find $\delta > 0$ such that, $|f(x) - l| < \in$ whenever $0 < |x-a| < \delta$ and 'l' is a finite real number.

We are going to study the limit of a rational function $\dfrac{P(x)}{Q(x)}$ as $x \to$ a.

Here P(x) and Q(x) are polynomials in x.

We get three different cases.

(1) $Q(a) \neq 0$,

(2) $Q(a) = 0$ and $P(a) = 0$

(3) $Q(a) = 0$ and $P(a) \neq 0$

In case (1) $\displaystyle\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$.

Because as $x \to a$, $P(x) \to P(a)$ and $Q(x) \to Q(a)$

In Case (2) $x - a$ is a factor of $P(x)$ as well as Q(x) so we have express $P(x)$ and $Q(x)$ as $P(x) = (x - a)\, P_1(x)$ and $Q(x) = (x - a)\, Q_1(x)$

Now $\dfrac{P(x)}{Q(x)} = \dfrac{(x-a)P_1(x)}{(x-a)Q_1(x)} = \dfrac{P_1(x)}{Q_1(x)}$.

Note that

$(x\text{-}a) \neq 0$ so we can cancel the factor.

In case (3) $Q(a) = 0$ and $P(a) \neq 0$,

$\lim\limits_{x \to a} \dfrac{P(x)}{Q(x)}$ does not exist.

7.1.1 One Sided Limit: You are aware of the fact that when $x \to a$; x approaches a in two directions; which can lead to two limits, known as left hand limit and right hand limit.

7.1.2 Right hand Limit : *If given $\in > 0$ (however small), there exists $\delta > 0$ such that $|f(x) - l_1| < \in$ for all x with $a < x < a + \delta$ then* $\lim\limits_{x \to a^+} f(x) = l_1$

7.1.3 Left hand Limit : *If given $\in > 0$ (however small), there exists $\delta > 0$ such that for $|f(x) - l_2| < \in$ all x with $a - \delta < x < a$ then* $\lim\limits_{x \to a^-} f(x) = l_2$

Example:

Find left hand limit and right hand limit for the following example.

$$f(x) = \begin{cases} 3x + 1 & if\ x < 1 \\ 7x^2 - 3 & if\ x \geq 1 \end{cases}$$

To compute, $\lim\limits_{x \to 1^+} f(x)$, we use the definition for f which applies to $x \geq 1$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (7x^2 - 3) = 4$$

Likewise, to compute $\lim\limits_{x \to 1^-} f(x)$, we use the definition for f which applies to $x < 1$:

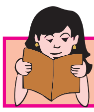
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x + 1) = 4$$

Since left and right-hand limits are equal,

$\lim\limits_{x \to 1} f(x) = 4$

7.1.4 Existence of a limit of a function at a point *x = a*

If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = l$, then limit of the function f(x) as $x \to a$ exists and its value is l. If $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$ then $\lim\limits_{x \to a} f(x)$ does not exist.



Let's learn.

7.2 ALGEBRA OF LIMITS:

Let f(x) and g(x) be two functions such that $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

1. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
   $= l \pm m$

2. $\lim\limits_{x \to a} [f(x) \times g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$
   $= l \times m$

3. $\lim\limits_{x \to a} [k\, f(x)] = k \lim\limits_{x \to a} f(x) = kl$, where '$k$' is a constant

4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$ provided $m \neq 0$

7.3 EVALUATION OF LIMITS :

Direct Method : In some cases $\lim\limits_{x \to a} f(x)$ can be obtained by just substitution of x by a in $f(x)$

SOLVED EXAMPLES

Ex. 1) $\lim_{r\to 1}\left(\frac{4}{3}\pi r^2\right)=\frac{4}{3}\pi\lim_{r\to 1}\left(r^2\right)=\frac{4}{3}\pi(1)^2=\frac{4}{3}\pi$

Ex. 2) $\lim_{y\to 2}[(y^2-3)(y+2)]$

$= \lim_{y\to 2}(y^2-3)\lim_{y\to 2}(y+2)$

$= (2^2-3)(2+2) = (4-3)(4) = 1\times 4 = 4$

Ex. 3) $\lim_{x\to 3}\left(\frac{\sqrt{6+x}-\sqrt{7-x}}{x}\right)$

$= \frac{\lim_{x\to 3}\left(\sqrt{6+x}\right)-\lim_{x\to 3}\left(\sqrt{7-x}\right)}{\lim_{x\to 3}(x)}$

$= \frac{\sqrt{6+3}-\sqrt{7-3}}{3}$

$= \frac{\sqrt{9}-\sqrt{4}}{3}$

$= \frac{3-2}{3}=\frac{1}{3}$

Ex. 4) Discuss the limit of the following function as x tends to 3 if

$$f(x)=\begin{cases}x^2+x+1, & 2\le x\le 3\\ 2x+1, & 3<x\le 4\end{cases}$$

Solution: we use the concept of left hand limit and right hand limit, to discuss the existence of limit as $x\to 3$

Note : *In both cases **x** takes only positive values.*

For the interval $2\le x\le 3$; $f(x) = x^2 + x + 1$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x^2+x+1)$

$= (3)^2 + 3 + 1$

$= 9 + 3 + 1 = 13$ -----(I)

Similarly for the interval $3 < x \le 4$;
$f(x) = 2x + 1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x+1) = (2\times 3) + 1 = 6 + 1$

$= 7$ -----(II)

From (I) and (II), $\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Theorem: $\lim_{x\to a}\frac{x^n-a^n}{x-a}=n(a^{n-1})$, for $n\in Q$.

SOLVED EXAMPLES

Ex. 1) $\lim_{x\to 4}\left[\frac{x^n-4^n}{x-4}\right]$ = 48 and $n\in N$, find n.

Solution: Given $\lim_{x\to 4}\left[\frac{x^n-4^n}{x-4}\right]=48$

$\therefore n(4)^{n-1} = 48 = 3\times 16 = 3(4)^2$

$\therefore n(4)^{n-1} = 3(4)^{3-1}$… by observation

$\therefore n = 3.$

Ex. 2) Evaluate $\lim_{x\to 1}\left[\frac{2x-2}{\sqrt[3]{26+x}-3}\right]$

Solution: Put $26 + x = t^3$, $\therefore x = t^3 - 26$
As $x\to 1$, $t\to 3$

$\therefore \lim_{x\to 1}\left[\frac{2x-2}{\sqrt[3]{26+x}-3}\right]$

$= \lim_{t\to 3}\left[\frac{2\left(t^3-26\right)-2}{\sqrt[3]{t^3}-3}\right]$

$= \lim_{1\to 3}\left[\frac{t^3-3^3}{t-3}\right]$

As $\lim_{x\to a}\left[\frac{x^n-a^n}{x-a}\right]=na^{n-1}$

$= 2 \times 3(3)^{3-1}$

$= 2 \times 3^3 = 2 \times 27$

$= 54$

EXERCISE 7.1

Q.I Evaluate the Following limits :

1. $\lim_{x \to 3} \left[\frac{\sqrt{x+6}}{x} \right]$

2. $\lim_{x \to 2} \left[\frac{x^{-3} - 2^{-3}}{x-2} \right]$

3. $\lim_{x \to 5} \left[\frac{x^3 - 125}{x^5 - 3125} \right]$

4. If $\lim_{x \to 1} \left[\frac{x^4 - 1}{x-1} \right] = \lim_{x \to a} \left[\frac{x^3 - a^3}{x-a} \right]$ find all possible values of a.

Q.II Evaluate the Following limits :

1. $\lim_{x \to 7} \left[\frac{\left(\sqrt[3]{x} - \sqrt[3]{7}\right)\left(\sqrt[3]{x} + \sqrt[3]{7}\right)}{x-7} \right]$

2. If $\lim_{x \to 5} \left[\frac{x^k - 5^k}{x-5} \right] = 500$ find all possible values of k.

3. $\lim_{x \to 0} \left[\frac{(1-x)^8 - 1}{(1-x)^2 - 1} \right]$

Q.III Evaluate the Following limits :

1. $\lim_{x \to 0} \left[\frac{\sqrt[3]{1+x} - \sqrt{1+x}}{x} \right]$

2. $\lim_{y \to 1} \left[\frac{2y-2}{\sqrt[3]{7+y} - 2} \right]$

3. $\lim_{z \to a} \left[\frac{(z+2)^{\frac{3}{2}} - (a+2)^{\frac{3}{2}}}{z-a} \right]$

4. $\lim_{x \to 5} \left[\frac{x^3 - 125}{x^2 - 25} \right]$

Let's learn.

7.4 FACTORIZATION METHOD :

Consider the problem of evaluating,

$\lim_{x \to a} \frac{f(x)}{g(x)}$ where $g(a) \neq 0$.

SOLVED EXAMPLES

Ex. 1) Evaluate $\lim_{z \to 3} \left[\frac{z(2z-3)-9}{z^2 - 4z + 3} \right]$

Solution: If we substitute z = 3 in numerator and denominator,

we get $z(2z - 3) - 9 = 0$ and $z^2 - 4z + 3 = 0$.

So $(z - 3)$ is a factor in the numerator and denominator.

$$\therefore \lim_{z \to 3} \left[\frac{z(2z-3)-9}{z^2 - 4z + 3} \right]$$

$$= \lim_{z \to 3} \left[\frac{2z^2 - 3z - 9}{z^2 - 4z + 3} \right]$$

$$= \lim_{z \to 3} \left[\frac{(z-3)(2z+3)}{(z-3)(z-1)} \right]$$

$$= \lim_{z \to 3} \left[\frac{(2z+3)}{(z-1)} \right] \quad \because (z-3 \neq 0)$$

$$= \frac{2(3)+3}{3-1}$$

$$= \frac{9}{2}.$$

Ex. 2) Evaluate $\lim_{x\to 4}\left[\frac{(x^3-8x^2+16x)^9}{\left(x^2-x-12\right)^{18}}\right]$

Solution : $\lim_{x\to 4}\left[\frac{\left[x\left(x-4\right)^2\right]^9}{\left(x-4\right)^{18}\left(x+3\right)^{18}}\right]$

$= \lim_{x\to 4}\left[\frac{\left(x-4\right)^{18}x^9}{\left(x-4\right)^{18}\left(x+3\right)^{18}}\right]$

$= \lim_{x\to 4}\left[\frac{x^9}{\left(x+3\right)^{18}}\right] \quad \because (x-4) \neq 0$

$= \frac{4^9}{7^{18}}$

Ex. 3) Evaluate $\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{2}{1-x^2}\right]$

Solution : $\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{2}{\left(1-x\right)\left(x+1\right)}\right]$

$= \lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{\left(x-1\right)\left(x+1\right)}\right]$

$= \lim_{x\to 1}\left[\frac{1+x-2}{\left(x-1\right)\left(x+1\right)}\right]$

$= \lim_{x\to 1}\left[\frac{x-1}{\left(x-1\right)\left(x+1\right)}\right]$

Since $(x \to 1)$, $(x - 1 \neq 0)$

$= \lim_{x\to 1}\left[\frac{1}{\left(x+1\right)}\right]$

$= \frac{1}{2}$

Ex. 4) Evaluate $\lim_{x\to 1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$

Solution : In this case $(x - 1)$ is a factor of the numerator and denominator.

To find another factor we use synthetic division, Numerator: $x^3 + x^2 - 5x + 3$

1	1	1	–5	3
		1	2	–3
	1	2 (=1+1)	–3(= –5+2)	0(= –3+3)

$\therefore x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3)$

Denominator: $x^2 - 1 = (x + 1)(x - 1)$

$= \lim_{x\to 1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$

$= \lim_{x\to 1}\left[\frac{\left(x-1\right)\left(x^2+2x-3\right)}{\left(x+1\right)\left(x-1\right)}\right]$

$= \lim_{x\to 1}\left[\frac{\left(x^2+2x-3\right)}{\left(x+1\right)}\right] \quad \left(\begin{array}{l} x\to 1 \\ x\neq 1 \\ x-1\neq 0 \end{array}\right)$

$= \frac{1+2-3}{1+1}$

$= 0$

Ex. 5) $\lim_{x\to 1}\left[\frac{\frac{1}{x}-1}{x-1}\right]$

$= \lim_{x\to 1}\left[\frac{\left(1-x\right)}{\left(x-1\right)\times x}\right]$

$= \lim_{x\to 1}\left[\frac{-\left(x-1\right)}{\left(x-1\right)\times x}\right]$

since $x - 1 \neq 0$

$= \lim_{x\to 1}\left[\frac{-1}{x}\right] = -\lim_{x\to 1}\left[\frac{1}{x}\right] = -\frac{1}{1}$

$= -1$

EXERCISE 7.2

Q.I Evaluate the following limits :

1. $\lim_{z\to 2}\left[\dfrac{z^2-5z+6}{z^2-4}\right]$

2. $\lim_{x\to -3}\left[\dfrac{x+3}{x^2+4x+3}\right]$

3. $\lim_{y\to 0}\left[\dfrac{5y^3+8y^2}{3y^4-16y^2}\right]$

4. $\lim_{x\to -2}\left[\dfrac{-2x-4}{x^3+2x^2}\right]$

Q.II Evaluate the following limits :

1. $\lim_{u\to 1}\left[\dfrac{u^4-1}{u^3-1}\right]$

2. $\lim_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{9x}{x^3-27}\right]$

3. $\lim_{x\to 2}\left[\dfrac{x^3-4x^2+4x}{x^2-1}\right]$

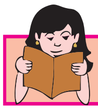
Q.III Evaluate the following limits :

1. $\lim_{x\to -2}\left[\dfrac{x^7+x^5+160}{x^3+8}\right]$

2. $\lim_{y\to \frac{1}{2}}\left[\dfrac{1-8y^3}{y-4y^3}\right]$

3. $\lim_{v\to \sqrt{2}}\left[\dfrac{v^2+v\sqrt{2}-4}{v^2-3v\sqrt{2}+4}\right]$

4. $\lim_{x\to 3}\left[\dfrac{x^2+2x-15}{x^2-5x+6}\right]$



Let's learn.

7.5 RATIONALIZATION METHOD :

If the function in the limit involves a square root, it may be possible to simplify the expression by multiplying and dividing by the conjugate. This method uses the algebraic identity.

Here, we do the following steps:
Step 1. **Rationalize the factor containing square root.**
Step 2. **Simplify.**
Step 3. **Put the value of *x* and get the required result.**

SOLVED EXAMPLES

Ex. 1) Evaluate $\lim_{z\to 0}\left[\dfrac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

Solution: $\lim_{z\to 0}\left[\dfrac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

$= \lim_{z\to 0}\left[\dfrac{\sqrt{b+z}-\sqrt{b-z}}{z}\right]$

$= \lim_{z\to 0}\left[\dfrac{\sqrt{b+z}-\sqrt{b-z}}{z}\times\dfrac{\sqrt{b+z}+\sqrt{b-z}}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\dfrac{\left(\sqrt{b+z}\right)^2-\left(\sqrt{b-z}\right)^2}{z}\times\dfrac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\dfrac{(b+z)-(b-z)}{z}\times\dfrac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\dfrac{2z}{z}\times\dfrac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$$= \lim_{z\to 0}\left[\frac{2}{\sqrt{b+z}+\sqrt{b-z}}\right]$$

$$= \frac{2}{\sqrt{b+0}+\sqrt{b-0}}$$

$$= \frac{2}{2\sqrt{b}}$$

$$= \frac{1}{\sqrt{b}}$$

Ex. 2) Evaluate $\lim_{x\to 0}\left[\frac{1-\sqrt{x^2+1}}{x^2}\right]$

Solution : $\lim_{x\to 0}\left[\frac{1-\sqrt{x^2+1}}{x^2}\right]$

$$= \lim_{x\to 0}\left[\frac{1-\sqrt{x^2+1}}{x^2}\frac{\left(1+\sqrt{x^2+1}\right)}{\left(1+\sqrt{x^2+1}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{1-\left(x^2+1\right)}{x^2}\frac{1}{\left[1+\sqrt{x^2+1}\right]}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(-x^2\right)}{x^2}\frac{1}{\left[1+\sqrt{x^2+1}\right]}\right]$$

$$= \lim_{x\to 0}\left[\frac{-1}{\left[1+\sqrt{x^2+1}\right]}\right]$$

$$= \frac{-1}{1+1}$$

$$= \frac{-1}{2}$$

EXERCISE 7.3

Q.I Evaluate the following limits :

1. $\lim_{x\to 0}\left[\frac{\sqrt{6+x+x^2}-\sqrt{6}}{x}\right]$

2. $\lim_{y\to 0}\left[\frac{\sqrt{1-y^2}-\sqrt{1+y^2}}{y^2}\right]$

3. $\lim_{x\to 2}\left[\frac{\sqrt{2+x}-\sqrt{6-x}}{\sqrt{x}-\sqrt{2}}\right]$

Q.II Evaluate the following limits :

1. $\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

2. $\lim_{x\to 2}\left[\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

Q.III Evaluate the Following limits :

1. $\lim_{x\to 1}\left[\frac{x^2+x\sqrt{x}-2}{x-1}\right]$

2. $\lim_{x\to 0}\left[\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\right]$

3. $\lim_{x\to 4}\left[\frac{x^2+x-20}{\sqrt{3x+4}-4}\right]$

4. $\lim_{x\to 2}\left[\frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

Q.IV Evaluate the Following limits :

1. $\lim_{y\to 2}\left[\frac{2-y}{\sqrt{3-y}-1}\right]$

2. $\lim_{z\to 4}\left[\frac{3-\sqrt{5+z}}{1-\sqrt{5-z}}\right]$

7.6 LIMITS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS :

We use the following results without proof.

1. $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$, a > 0,

2. $\lim_{x\to 0}\frac{e^x-1}{x}=\log e=1$

3. $\lim_{x\to 0}[1+x]^{\frac{1}{x}}=e$

4. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

SOLVED EXAMPLES

Ex.1) Evalute : $\lim_{x\to 0}\left[\frac{7^x-1}{x}\right]$

Solution : $\lim_{x\to 0}\left[\frac{7^x-1}{x}\right]$

$= \log 7$

Ex.2) Evaluate : $\lim_{x\to 0}\left[\frac{5^x-3^x}{x}\right]$

Solutions : $\lim_{x\to 0}\left[\frac{5^x-3^x}{x}\right]$

$=\lim_{x\to 0}\left[\frac{5^x-1-3^x+1}{x}\right]$

$=\lim_{x\to 0}\left[\frac{(5^x-1)}{x}-\frac{(3^x-1)}{x}\right]$

$=\lim_{x\to 0}\left[\frac{(5^x-1)}{x}\right]-\lim_{x\to 0}\left[\frac{(3^x-1)}{x}\right]$

$= \log 5 - \log 3$

$= \log\left(\frac{5}{3}\right)$

Ex.3) Evaluate : $\lim_{x\to 0}\left[1+\frac{5}{6}x\right]^{\frac{1}{x}}$

Solutions : $\lim_{x\to 0}\left[1+\frac{5}{6}x\right]^{\frac{1}{x}}$

$=\lim_{x\to 0}\left[\left(1+\frac{5}{6}x\right)^{\frac{1}{\frac{5}{6}x}}\right]^{\frac{5}{6}}$

$= e^{\frac{5}{6}}$

Ex.4) Evaluate : $\lim_{x\to 0}\left[\frac{\log(1+4x)}{x}\right]$

Solutions : $\lim_{x\to 0}\left[\frac{\log(1+4x)}{x}\right]$

$=\lim_{x\to 0}\left[\frac{\log(1+4x)}{4x}\times 4\right]$

$= 4 \times 1$

$= 4$

Ex.5) Evaluate :

$\lim_{x\to 0}\left[\frac{8^x-4^x-2^x+1}{x^2}\right]$

Solutions : Given $\lim_{x\to 0}\left[\frac{8^x-4^x-2^x+1}{x^2}\right]$

$=\lim_{x\to 0}\left[\frac{(4\times 2)^x-4^x-2^x+1}{x^2}\right]$

$=\lim_{x\to 0}\left[\frac{4^x.2^x-4^x-2^x+1}{x^2}\right]$

$=\lim_{x\to 0}\left[\frac{4^x\left(2^x-1\right)-(2^x-1)}{x^2}\right]$

$$= \lim_{x\to 0}\left[\frac{\left(2^x-1\right).\left(4^x-1\right)}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(2^x-1\right)}{x}\right] \times \lim_{x\to 0}\left[\frac{\left(4^x-1\right)}{x}\right]$$

$$= (log2)\ (log4)$$

EXERCISE 7.4

I] Evaluate the following :

1) $\lim_{x\to 0}\left[\frac{9^x-5^x}{4^x-1}\right]$

2) $\lim_{x\to 0}\left[\frac{5^x+3^x-2^x-1}{x}\right]$

3) $\lim_{x\to 0}\left[\frac{log\left(2+x\right)-log\left(2-x\right)}{x}\right]$

II] Evaluate the following :

1) $\lim_{x\to 0}\left[\frac{3^x+3^{-x}-2}{x^2}\right]$

2) $\lim_{x\to 0}\left[\frac{3+x}{3-x}\right]^{\frac{1}{x}}$

3) $\lim_{x\to 0}\left[\frac{log\left(3-x\right)-log\left(3+x\right)}{x}\right]$

III] Evaluate the following :

1) $\lim_{x\to 0}\left[\frac{a^{3x}-b^{2x}}{\log(1+4x)}\right]$

2) $\lim_{x\to 0}\left[\frac{\left(2^x-1\right)^2}{(3^x-1).\log(1+x)}\right]$

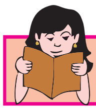
3) $\lim_{x\to 0}\left[\frac{15^x-5^x-3^x+1}{x^2}\right]$

4) $\lim_{x\to 2}\left[\frac{3^{\frac{x}{2}}-3}{3^x-9}\right]$

IV] Evaluate the following :

1) $\lim_{x\to 0}\left[\frac{(25)^x-2\left(5\right)^x+1}{x^2}\right]$

2) $\lim_{x\to 0}\left[\frac{(49)^x-2\left(35\right)^x+\left(25\right)^x}{x^2}\right]$



Let's learn.

Some Standard Results

1. $\lim_{x\to a} k = k$, where k is a constant

2. $\lim_{x\to a}\ x = a$

3. $\lim_{x\to a}\ x^n = a^n$

4. $\lim_{x\to a}\sqrt[r]{x} = \sqrt[r]{a}$

5. If p(x) is a polynomial, then $\lim_{x\to a}\ p(x) = p(a)$

6. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = n(a^{n-1})$, for $n\in Q$

7. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = \log e = 1$

8. $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

MISCELLANEOUS EXERCISE - 7

I. If $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$ then find the value of n.

II. Evaluate the following Limits.

1) $\lim_{x\to a}\frac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a}$

2) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$

3) $\lim\limits_{x\to 2}\left[\dfrac{(x-2)}{2x^2-7x+6}\right]$

4) $\lim\limits_{x\to 1}\left[\dfrac{x^3-1}{x^2+5x-6}\right]$

5) $\lim\limits_{x\to 3}\left[\dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}\right]$

6) $\lim\limits_{x\to 4}\left[\dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right]$

7) $\lim\limits_{x\to 0}\left[\dfrac{5^x-1}{x}\right]$

8) $\lim\limits_{x\to 0}\left(1+\dfrac{x}{5}\right)^{\frac{1}{x}}$

9) $\lim\limits_{x\to 0}\left[\dfrac{\log(1+9x)}{x}\right]$

10) $\lim\limits_{x\to 0}\dfrac{(1-x)^5-1}{(1-x)^3-1}$

11) $\lim\limits_{x\to 0}\left[\dfrac{a^x+b^x+c^x-3}{x}\right]$

12) $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{x^2}$

13) $\lim\limits_{x\to 0}\left[\dfrac{x(6^x-3^x)}{(2^x-1).\log(1+x)}\right]$

14) $\lim\limits_{x\to 0}\left[\dfrac{a^{3x}-a^{2x}-a^x+1}{x^2}\right]$

15) $\lim\limits_{x\to 0}\left[\dfrac{(5^x-1)^2}{x.\log(1+x)}\right]$

16) $\lim\limits_{x\to 0}\left[\dfrac{a^{4x}-1}{b^{2x}-1}\right]$

17) $\lim\limits_{x\to 0}\left[\dfrac{log100+\log(0.01+x)}{x}\right]$

18) $\lim\limits_{x\to 0}\left[\dfrac{\log(4-x)-\log(4+x)}{x}\right]$

19) Evaluate the limit of the function if exist at $x = 1$ where $f(x) = \begin{cases} 7-4x & x<1 \\ x^2+2 & x\geq 1 \end{cases}$

Activity 7.1

Evaluate : $\lim\limits_{x\to 0}\left[\dfrac{e^x-x-1}{x}\right]$

Solution : $= \lim\limits_{x\to 0}\left[\dfrac{(e^x-1)-\square}{x}\right]$

$= \lim\limits_{x\to 0}\dfrac{\square}{x}-\dfrac{x}{x}$

$= \lim\limits_{x\to 0}\left[\dfrac{e^x-1}{x}\right]-\square$

$= \square - 1$

$= \square$

Activity 7.2

Carry out the following activity.

Evaluate: $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}+\dfrac{2}{1-x^2}\right]$

Solution : $\lim\limits_{x\to 1}\left[\dfrac{1}{x-1}-\dfrac{2}{\square}\right]$

$= \lim\limits_{x\to 1}\left[\dfrac{\square}{(x-1)(x+1)}\right]$

$= \lim\limits_{x\to 1}\dfrac{1}{x+1}$

$= \square$

❖ ❖ ❖

8. CONTINUITY

Let's study.

- Continuity of a function at a point.
- Continuity of a function over an interval.

Let's recall.

- Different types of functions.
- Limits of Algebraic, Exponential, Logarithmic functions.
- Left hand and Right hand limits of functions.

Let's learn.

8.1 CONTINUOUS AND DISCONTINUOUS FUNCTIONS

The dictionary meaning of the word continuity **is the unbroken and consistent existence over a period of time.** The intuitive idea of continuity is manifested in the following examples.

(i) An unbroken road between two cities.

(ii) Flow of river water.

(iii) The story of a drama.

(iv) Railway tracks.

(v) Temperature of a city on a day changing with time.

The temperature of Pune rises from 14^0 C at night to 29^0 C in the afternoon, this change in the temperature is continuous and all the values between 14 and 29 are taken during 12 hours. An activity that takes place gradually, without interruption or abrupt change is called a continuous process. Similarly to ensure continuity of a function, there should not be any interruption or jump, or break in the graph of a function.

8.2 CONTINUITY AT A POINT

We are going to study continuity of functions of real variable. So the domain will be an interval in R. Before we consider a formal definition of continuity of a function at a point, let's consider various functions .

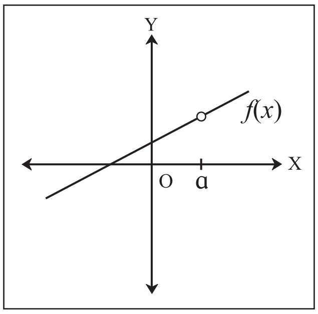


Figure 8.1

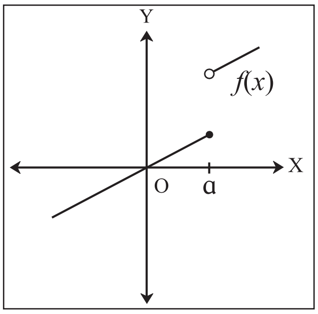


Figure 8.2

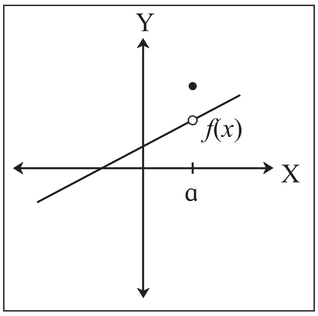


Figure 8.3

In Figure 8.1, we see that the graph of $f(x)$ has a hole at $x = a$. In fact, $f(a)$ is not defined for $x = a$. At the very least, for $f(x)$ to be continuous at $x = a$, we need $f(a)$ to be defined.

Condition 1 : If $f(x)$ is to be continuous at $x = a$ then $f(a)$ must be defined.

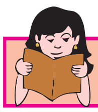
Now, let us see Figure 8.2. Although $f(a)$ is defined, the graph has a gap at $x = a$. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. Because, left hand limit at x = a is not equal to its right hand limit at x = a. So, we must add another condition for continuity at $x = a$, will be that $\lim_{x \to a} f(x)$ must exist. So the right hand limit and left hand limits are equal.

Condition 2 : If f(x) is to be continuous at $x = a$ then $\lim_{x \to a} f(x)$ must exist.

In Figure 8.3, The function in this figure satisfies both of our first two conditions, but is still there is a hole in the graph of the function. We must add a third condition that $\lim_{x \to a} f(x) = f(a)$.

Condition 3 : If f(x) is to be continuous at $x = a$ then $\lim_{x \to a} f(x) = f(a)$.

Now we put our list of conditions together and form the definition of continuity at a point.



Let's learn.

8.3 DEFINITION OF CONTINUITY

A function $f(x)$ is said to be continuous at point $x = a$ if the following three conditions are satisfied:

i. f is defined on an open interval containing a.

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$.

The condition (iii) can be reformulated and the continuity of $f(x)$ at $x = a$, can be restated as follows :

A function $f(x)$ is said to be continuous at a point $x = a$ if it is defined in some neighborhood of 'a' and if $\lim_{h \to 0} [f(a+h) - f(a)] = 0$.

Discontinuous Function : A function $f(x)$ is not continuous at $x = c$, it is said to be discontinuous at $x = c$, 'c' is called the point of discontinuity.

Example 1. Consider $f(x) = |x|$ be defined on R.

$$f(x) = -x \quad \text{for } x < 0$$
$$= x \quad \text{for } x \geq 0 .$$

Let us discuss the continuity of $f(x)$ at $x = 0$

Consider, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$$

$$\text{For} \quad x = 0, \quad f(0) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

Hence $f(x)$ is continuous at $x = 0$.

Example 2 :

Consider $f(x) = x^2$. Let us discuss the continuity of f at $x = 2$.

$$f(x) = x^2$$
$$f(2) = 2^2 = 4$$
$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2) = 2^2 = 4$$
$$\lim_{x \to 2} f(x) = f(2) = 4$$

Therefore the function $f(x)$ is continuous at x=2.

There are some functions, which are defined in two different ways on either side of a point. In such cases we have to consider the limits of function from left as well as from right of that point.

8.4 CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f(x)$ is said to be continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

Consider the following examples.

Example 1: Let us discuss the continuity of $f(x) = [x]$ in the interval [2, 4)

[Note : [x] is the greatest integer function or floor function]

$f(x) = [x]$, for $x \in [2, 4)$

that is $f(x) = 2$, for $x \in [2, 3)$

$f(x) = 3$, for $x \in [3, 4)$

The graph of the function is as shown below,

Test of continuity at $x = 3$.

For $x = 3$, $f(3) = 3$

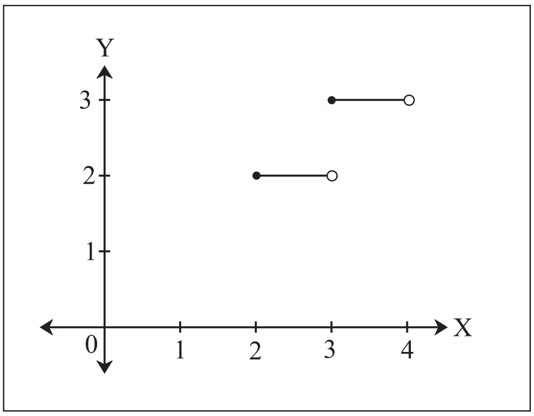


Figure 8.4

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} [x] = 2 \text{ and}$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} [x] = \lim_{x \to 3^+} (3) = 3$$

$$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 3$.

Example 2 : Let us discuss the continuity of

$f(x) = x^2 + 2$ for $0 \leq x \leq 2$

$= 5x - 4$ for $2 < x \leq 3.5$

For $x = 2$, $f(2) = 2^2 + 2 = 6$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 2) = (2^2+2) = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4) = (10-4) = 6$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 6 \Rightarrow \lim_{x \to 2} f(x) = f(2)$$

Therefore $f(x)$ is continuous at $x = 2$.

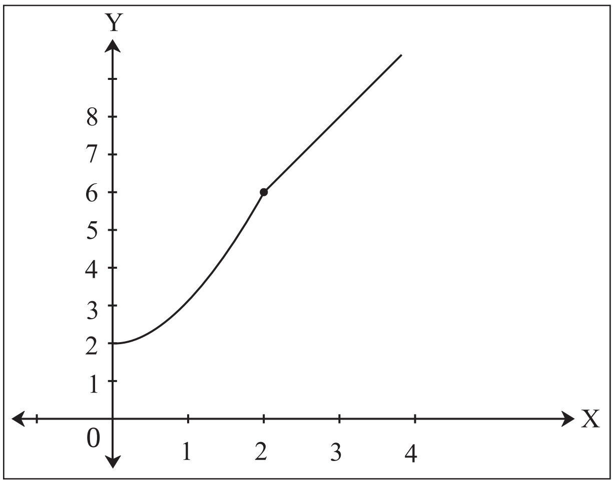


Figure 8.5

8.5 PROPERTIES OF CONTINUOUS FUNCTIONS:

If the functions f and g are continuous at $x = a$, then,

1. their sum $(f + g)$ is continuous at $x = a$.
2. their difference $(f - g)$ is continuous at $x = a$.
3. constant multiples that is $k.f$ for any $k \in R$ is continuous at $x = a$.
4. their product that is $(f.g)$ is continuous at $x = a$.

5. their quotient that is $\frac{f}{g}$ where $g(a) \neq 0$ is continuous at $x = a$.

6. their composite function $f\ [g(x)]$, that is $fog\ (x)$ is continuous at $x = a$.

8.6 CONTINUITY OVER AN INTERVAL

So far we have explored the concept of continuity of a function at a point. Now we will extend the idea of continuity to an interval.

Let (a, b) be an open interval. If for every

$x \in (a, b)$, f is continuous at x then we say that f is continuous on (a, b) .

Consider f on $[a, b)$ if f is continuous on (a, b) and f is continuous to the right of a, then f is continuous on $[a, b)$

Consider f on $(a, b]$ if f is continuous on

(a, b) and f is continuous to the left of b, then f is continuous on $(a, b]$

Consider a function f continuous on the open interval (a,b). If $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$ exists, then we can extend the function to $[a,b]$ so that it is continuous on $[a,b]$.

Definition : A real valued function f is said to be continuous in an interval if it is continuous at every point of the interval.

8.7 CONTINUITY IN THE DOMAIN OF THE FUNCTION :

A real valued function $f : D \to R$ is said to be continuous if it is continuous at every point in the domain D of 'f'.

SOLVED EXAMPLES

Example 1 : Determine whether the function f is discontinuous for any real numbers in its domain.

where $f(x) = 3x + 1$, for $x < 2$

$= 7$, for $2 \le x < 4$

$= x^2 - 8$ for $x \ge 4$.

Solution :

Let us check the conditions of continuity at $x = 2$.

For $x = 2$, $f(2) = 7$ (Given)

[Condition 1 is satisfied]

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3x+1) = 3(2) + 1 = 7.$$

and $\lim_{x\to 2^+} f(x) = 7$,

So $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 7 \Rightarrow \lim_{x\to 2} f(x) = 7$

[Condition 2 is satisfied]

Also, $\lim_{x\to 2} f(x) = 7 = f(2)$, $\lim_{x\to 2} f(x) = f(2)$

[Condition 3 is satisfied]

Therefore $f(x)$ is continuous at $x = 2$.

Let us check the conditions of continuity at $x = 4$.

For $x = 4$, $f(4) = (4^2 - 8) = 8$

[Condition 1 is satisfied]

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (7) = 7 \text{ and}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x^2 - 8) = 4^2 - 8 = 8$$

$\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$ So, $\lim_{x\to 4} f(x)$ does not exist.

[Condition 2 is not satisfied]

Since one of the three conditions is not satisfied at $x = 4$, the function $f(x)$ is discontinuous at $x = 4$. Therefore the function f is continuous on it's domain, except at $x = 4$.

Example 2 : Test whether the function $f(x)$ is continuous at $x = -4$, where

$$f(x) = \frac{x^2+16x+48}{x+4} \quad \text{for } x \neq -4$$

$$= 8 \quad \text{for } x = -4.$$

Solution : For $x = -4$, $f(-4) = 8$ (defined)

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \left(\frac{x^2+16x+48}{x+4}\right)$$

$$= \lim_{x \to -4} \left(\frac{(x+4)(x+12)}{x+4} \right)$$

$$\lim_{x \to -4} (x+12) = -4 + 12 = 8$$

$$\therefore \lim_{x \to -4} f(x) \text{ exists}$$

$$\therefore \lim_{x \to -4} f(x) = f(-4) = 8$$

Therefore the function $f(x)$ is continuous at $x = -4$.

Example 3 : If $f(x) = \left(\frac{3x+2}{2-5x} \right)^{\frac{1}{x}}$ for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$.

Solution : Given that $f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x \to 0} f(x)$$

$$f(0) = \lim_{x \to 0} \left(\frac{3x+2}{2-5x} \right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \left(\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)} \right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \left(\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}} \right)$$

$$= \frac{\left[\lim_{x \to 0} \left(1+\frac{3x}{2}\right)^{\frac{2}{3x}} \right]^{\frac{3}{2}}}{\left[\lim_{x \to 0} \left(1-\frac{5x}{2}\right)^{\frac{-2}{5x}} \right]^{\frac{-5}{2}}}$$

$$= \frac{e^{\frac{3}{2}}}{e^{\frac{-5}{2}}}$$

$$= e^{\frac{3}{2}+\frac{5}{2}} = e^{\frac{8}{2}} = e^4 \quad \because \left[\lim_{x \to 0} (1+kx)^{\frac{1}{kx}} = e \right]$$

8.8 EXAMPLES OF CONTINUOUS FUNCTIONS WHEREVER THEY ARE DEFINED:

(1) Constant function is continuous at every point of R.

(2) Power functions with positive integer exponents are continuous at every point of R.

(3) Polynomial functions,

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ are continuous at every point of R

(4) The exponential function a^x and logarithmic function $\log_b x$ (for any $x > 0$, base $b > 0$ and $b \neq 1$) are continuous for all $x \in R$

(5) Rational functions which are quotients of polynomials of the form $\frac{P(x)}{Q(x)}$ are continuous at every point , where $Q(x) \neq 0$.

(6) The n^{th} root functions, are continuous in their respective domains.

EXERCISE 8.1

1. Examine the continuity of

(i) $f(x) = x^3 + 2x^2 - x - 2$ at $x = -2$.

(ii) $f(x) = \frac{x^2 - 9}{x - 3}$ on R

2. Examine whether the function is continuous at the points indicated against them.

(i) $f(x) = x^3 - 2x + 1$, for $x \leq 2$

$= 3x - 2$, for $x > 2$, at $x = 2$.

(ii) $f(x) = \dfrac{x^2+18x-19}{x-1}$ for $x \neq 1$

$= 20$ for $x = 1$, at $x = 1$

3. Test the continuity of the following functions at the points indicated against them.

(i) $f(x) = \dfrac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$ for $x \neq 2$

$= \dfrac{1}{5}$ for $x = 2$, at $x = 2$

(ii) $f(x) = \dfrac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$, at $x = 2$

(iii) $f(x) = 4x + 1,$ for $x \leq 3$

$= \dfrac{59-9x}{3}$, for $x > 3$ at $x = \dfrac{8}{3}$.

(iv) $f(x) = \dfrac{x^3-27}{x^2-9}$ for $0 \leq x < 3$

$= \dfrac{9}{2}$ for $3 \leq x \leq 6$

at $x = 3$

4) (i) If $f(x) = \dfrac{24^x-8^x-3^x+1}{12^x-4^x-3^x+1}$ for $x \neq 0$

$= k,$ for $x = 0$

is continuous at $x = 0$, find k.

(ii) If $f(x) = \dfrac{5^x+5^{-x}-2}{x^2}$ for $x \neq 0$

$= k$ for $x = 0$

is continuous at $x = 0$, find k

(iii) For what values of a and b is the function

$f(x) = ax + 2b + 18$ for $x \leq 0$

$= x^2 + 3a - b$ for $0 < x \leq 2$

$= 8x - 2$ for $x > 2$,

continuous for every x ?

(iv) For what values of a and b is the function

$f(x) = \dfrac{x^2-4}{x-2}$ for $x < 2$

$= ax^2 - bx + 3$ for $2 \leq x < 3$

$= 2x - a + b$ for $x \geq 3$

continuous in its domain.

Let's remember!

Continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied:

(1) $f(a)$ is defined, on an open interval containing a

(2) $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$

Continuity from right : A function is continuous from right at a if $\lim_{x \to a^+} f(x) = f(a)$

Continuity from left : A function is continuous from left at b if $\lim_{x \to b^-} f(x) = f(b)$

Continuity over an interval :

Open Interval : A function is continuous over an open interval if it is continuous at every point in the interval.

Closed Interval : A function $f(x)$ is continuous over a closed interval of the form $[a,b]$ if it is continuous at every point in (a,b), and it is continuous from the right at a and continuous from the left at b

Discontinuity at a point : A function is discontinuous at a point if it is not continuous at that point.

MISCELLANEOUS EXERCISE - 8

(I) Discuss the continuity of the following functions at the point(s) or in the interval indicated against them.

1) $f(x) = 2x^2 - 2x + 5$ for $0 \le x < 2$

$= \dfrac{1-3x-x^2}{1-x}$ for $2 \le x < 4$

$= \dfrac{7-x^2}{x-5}$ for $4 \le x \le 7$ on its domain.

2) $f(x) = \dfrac{3^x+3^{-x}-2}{x^2}$ for $x \ne 0$.

$= (\log 3)^2$ for $x = 0$ at $x = 0$

3) $f(x) = \dfrac{5^x - e^x}{2x}$ for $x \ne 0$

$= \dfrac{1}{2}(\log 5 - 1)$ for $x = 0$ at $x = 0$.

4) $f(x) = \dfrac{\sqrt{x+3}-2}{x^3-1}$ for $x \ne 1$

$= 2$ for $x = 1$, at $x = 1$.

5) $f(x) = \dfrac{\log x - \log 3}{x-3}$ for $x \ne 3$

$= 3$ for $x = 3$, at $x = 3$.

(II) Find k if following functions are continuous at the points indicated against them.

(1) $f(x) = \left(\dfrac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$ for $x \ne 2$

$= k$ for $x = 2$ at $x = 2$.

(2) $f(x) = \dfrac{45^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)}$ for $x \ne 0$

$= \dfrac{2}{3}$ for $x = 0$, at $x = 0$

(3) $f(x) = (1+kx)^{\frac{1}{x}}$, for $x \ne 0$

$= e^{\frac{3}{2}}$, for $x = 0$, at $x = 0$

(III) Find a and b if following functions are continuous at the point indicated against them.

(1) $f(x) = x^2 + a$, for $x \ge 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$ and

$f(1) = 2$ is continuous at $x = 0$

(2) $f(x) = \dfrac{x^2-9}{x-3} + a$, for $x > 3$

$= 5$, $x = 3$

$= 2x^2 + 3x + b$, for $x < 3$

is continuous at $x = 3$

(3) $f(x) = \dfrac{32^x - 1}{8^x - 1} + a$, for $x > 0$

$= 2$, for $x = 0$

$= x + 5 - 2b$, for $x < 0$

is continuous at $x = 0$

ACTIVITIES

Activity 8.1:

If the following function is continuous at $x = 0$, find a and b.

$f(x) = x^2 + a$, for $x > 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$

$= 2$, for $x = 0$

Solution : Given

$f(x) = x^2 + a$, for $x > 0$

$= 2\sqrt{x^2+1} + b$, for $x < 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + a)$

$= (\square^2 + a)$

$\lim_{x\to 0^+} f(x) = \square$ .... (1)

$f(0) = 2$ .... (2)

Since $f(x)$ is continuous at $x = 0$

$\therefore \lim_{x\to 0^+} f(x) = f(0)$

$\therefore a = \square$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2\sqrt{x^2+1} + b)$

$= 2\sqrt{\square + 1} + b$

$\lim_{x\to 0^-} f(x) = \square + b$ .... (3)

Since f (x) is continuous at $x = 0$

$\lim_{x\to 0^-} f(x) = f(0)$

$\therefore \square + b = 2$

$\therefore b = \square$

Activity 8.2:

If $f(x) = \dfrac{x^2-4}{x-2}$, for $x \neq 2$ is continuous at $x = 2$ then find $f(2)$.

Solution : Given function,

$$f(x) = \frac{x^2-4}{x-2}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{x^2 - \square}{x - \square}$$

$$= \lim_{x\to 2} \frac{(x+\square)(x-\square)}{(x-\square)}$$

$$= \lim_{x\to 2} (x+\square)$$

$$= \square + \square$$

$\lim_{x\to 2} f(x) = \square$

Since $f(x)$ is continuous at $x = 2$

$\lim_{x\to 2} f(x) = f(2)$

$\lim_{x\to 2} f(x) = \square$

$\therefore f(2) = \square$

Activity 8.3:

Determine whether the function ' f ' is continuous on its domain.

$$f(x) = 3x + 1, \quad x < 2$$
$$= 7, \quad 2 \le x < 4$$
$$= x^2 - 8, \quad x \ge 4$$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \square$

$= \square$

Now $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \square = \square$

But $f(x) = \square$ at $x = 2$

$\therefore$ The function is $\square$ at $x = 2$

Also $\lim_{x\to 4^-} f(x) = \square$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \square = \square$

$\therefore$ Given function is $\square$ at $x = 4$

$\therefore$ the function is continuous at $x = \square$ and discontinuous at $x = \square$ in the domain of $f(x)$.

❖ ❖ ❖

9. DIFFERENTIATION

Let's study.

- The meaning of rate of change.
- Meaning of derivative and the formula associated with it.
- Derivatives of some standard functions.
- Some applications of derivatives

Let's recall.

- Real valued functions on R.
- Limits of functions.
- Continuity of a function at a point and over an interval.

The meaning of rate of change.

Suppose we are traveling in a car from Mumbai to Pune. We are displacing ourselves from the origin (Mumbai) from time to time. We know that the speed of the car

$$= \frac{\text{Distance travelled by the car}}{\text{Time taken to travel the distance}}$$

But at different times the speed of the car can be different. It is the ratio of, a very small distance travelled and very small time taken to travel that distance. The limit of this ratio, as the time interval tends to zero is the speed of the car at that time. This process of obtaining the speed is given by the differentiation of the distance function with respect to time. This is an example of derivative or differentiation which is useful. This measures how quickly the position of the object changes with time.

When we speak of velocity, it is the speed with the direction of movement. In problems with no change in direction, words speed and velocity may be interchanged. The rate of change in a function at a point with respect to the variable is called the derivative of the function at that point. The process of finding a derivative is called differentiation

9.1 Definition of Derivative and differentiability

Let f be a function defined on an open interval containing 'a'. If the limit $\lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}$

exists, then f is said to be differentiable at $x = a$. This limit is denoted by $f'(a)$ and is given by

$f'(a) = \lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x}$ $f'(a)$ is also called the derivative of 'f' at a.

If $y = f(x)$ is the functional and if the limiting value of the function exists, then that limiting value is called the derivative of the function and it is symbolically represented as,

$$\frac{dy}{dx} = f'(x) = y'$$

Let's Note.

(1) If y is a differentiable function of x then

$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx}$ and

$$\lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right) = f'(x) = y' = y_1$$

9.2 Derivative by the method of first principle.

The process of finding the derivative of a function using the definition of derivative is known as derivative from the first principle. Just for the sake of convenience δx is replaced by h.

If $y = f(x)$ is the given function, then the derivative of y *w.r.t.x* is represented as

$$\frac{dy}{dx} = f'(x) = \lim_{h \to o} \left(\frac{f(x+h) - f(x)}{h} \right)$$

9.3 Notations for derivatives :

If $y = f(x)$ is a given function then the derivative of $f(x)$ is represented in many different ways. Many mathematicians have used different notations for derivatives.

Most commonly used is the Leibniz's notation. The symbols dx, dy and $\frac{dy}{dx}$ were introduced by **Gottfried Wilhelm Leibnitz**.

Note : If we consider the graph of a function in XY-plane, then the derivative of the function f at a point 'a' , is the slope of the tangent line to the curve $y = f(x)$ at $x = a$. (We are going to study this in detail in next level)

Let's learn.

Derivatives of some standard functions

Ex. (1) Find the derivative of x^n w. r. t. x. $(n \in N)$

Solution :

Let $f(x) = x^n$

$f(x + h) = (x + h)^n$

By the method of first principle,

$$f'(x) = \lim_{h \to o} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(x+h)^n - x^n}{x+h-x} \right)$$

Let $x + h = y$, as $h \to 0$, $x \to y$

$$f'(x) = \lim_{x \to y} \left(\frac{y^n - x^n}{y - x} \right)$$

$$f'(x) = n.x^{n-1} \quad .... \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = n.a^{n-1} \right]$$

Ex. (2) Find the derivative of a^x w. r. t. $x.(a > 0)$

Solution :

Let $f(x) = a^x$

$f(x + h) = a^{x+h}$

By the method of first principle,

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{a^{(x+h)} - a^x}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{a^x(a^h - 1)}{h} \right)$$

$$= a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right)$$

$$f'(x) = a^x \log a \quad .... \left[\because \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \right]$$

Try the following

(1) If $f(x) = \frac{1}{x^n}$, then prove that $f'(x) = \frac{-n}{x^{n+1}}$

(2) If $f(x) = e^x$, then prove that $f'(x) = e^x$

SOLVED EXAMPLES

EX. 1. Find the derivatives of the following by using the method of first principle

(i) $\sqrt{x}$ (ii) 4^x (iii) $\log x$

Solution :

(i) Let $f(x) = \sqrt{x}$

$f(x+h) = \sqrt{x+h}$

By the method of first principle,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)}-\sqrt{x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)}-\sqrt{x}}{h}\times\frac{\sqrt{(x+h)}+\sqrt{x}}{\sqrt{(x+h)}+\sqrt{x}}\right)$$

$$= \lim_{h\to 0}\left(\frac{x+h-x}{h\left(\sqrt{(x+h)}+\sqrt{x}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{h}{h\left(\sqrt{(x+h)}+\sqrt{x}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right) \text{ .... (As } h\to 0, h\neq 0)$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

(ii) Let $f(x) = 4^x$

$f(x+h) = 4^{x+h}$

By method of first principle,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{4^{x+h}-4^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4^x(4^h-1)}{h}\right)$$

$$= 4^x\lim_{h\to 0}\left(\frac{4^h-1}{h}\right)$$

$$f'(x) = 4^x\log 4 \text{ ....}\left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

(iii) Let $f(x) = \log x$

$f(x+h) = \log(x+h)$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\log(x+h)-\log(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\log\left(\frac{x+h}{x}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\log\left(1+\frac{h}{x}\right)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right]\times\frac{1}{x}$$

$$= (1)\left(\frac{1}{x}\right)\text{....} \because \lim_{x\to 0}\left[\frac{\log(1+x)}{x}\right]=1$$

$$= \frac{1}{x}$$

$$\therefore f'(x) = \frac{1}{x}$$

EX. 2. Find the derivative of $x^2 + x + 2$ at $x = -3$

Let $f(x) = x^2 + x + 2$

For $x = -3$, $f(-3) = (-3)^2 - 3 + 2 = 9 - 3 + 2 = 8$

$f(-3+h) = (-3+h)^2 + (-3+h) + 2$
$= h^2 - 6h + 9 - 3 + h + 2 = h^2 - 5h + 8$

By method of first principle,

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$$\therefore\ f'(-3) = \lim_{h\to 0}\left(\frac{f(-3+h)-f(-3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h^2-5h+8-8}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(h-5)}{h}\right)$$

$$= \lim_{x\to 0}(h-5) = -5 \text{ (As } h\to 0, h\neq 0)$$

$$\therefore\ f'(-3) = -5$$

9.4 Rules of Differentiation (without proof)

Theorem 1. Derivative of Sum of functions.

If u and v are differentiable functions of x such that $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

Theorem 2. Derivative of Difference of functions.

If u and v are differentiable functions of x such that $y = u - v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

Corollary 1 : If $u,v,w....$ are differentiable functions of x such that $y = k_1u \pm k_2v \pm k_3w \pm ....$ where k_1, k_2, k_3 .... are constants then

$$\frac{dy}{dx} = k_1\frac{du}{dx} \pm k_2\frac{dv}{dx} \pm k_3\frac{dw}{dx} \quad ....$$

Theorem 3. Derivative of Product of functions.

If u and v are differentiable functions of x such that $y = u.v$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Corollary 2 : If u,v and w are differentiable functions of x such that $y = u.v.w$ then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Theorem 4. Derivative of Quotient of functions.

If u and v are differentiable functions of x such that $y = \dfrac{u}{v}$ where $v \neq 0$ then $\dfrac{dy}{dx} = \dfrac{v.\dfrac{du}{dx} - u.\dfrac{dv}{dx}}{v^2}$

Derivatives of Algebraic Functions

Sr. No.	y	dy/dx
	$f(x)$	$f'(x)$
01	x^n	nx^{n-1}
02	$\dfrac{1}{x^n}$	$-\dfrac{n}{x^{n+1}}$
03	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
04	c	0

SOLVED EXAMPLES

Ex. 1: Differentiate the following functions w.r.t.x.

i) $y = x^4 - 2x^3 + \sqrt{x} - \dfrac{3}{x^2} - 8$

ii) $y = (2x + 3)(x^3 - 7x + 4)$

iii) $x \log x$

iv) $y = \log x\,(e^x + x)$

Solution :

1) $y = x^4 - 2x^3 + \sqrt{x} - \dfrac{3}{x^2} - 8$

$\therefore\ y = x^4 - 2x^3 + x^{1/2} - 3x^{-2} - 8$

Differentiate *w.r.t.x*.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^{1/2})$$

$$-3\frac{d}{dx}(x^{-2}) - \frac{d}{dx}(8) \text{ .... (by rule 1)}$$

$$= 4x^3 - 2(3x^2) + (\frac{1}{2}x^{-1/2}) - 3(-2x^{-3}) - 0$$

$$= 4x^3 - 6x^2 + (\frac{1}{2}x^{-1/2}) + 6x^{-3}$$

$$= 4x^3 - 6x^2 + \frac{1}{2\sqrt{x}} + \frac{6}{x^3}$$

ii) $y = (2x + 3)(x^3 - 7x + 4)$

Differentiating *w.r.t.x*

$$\frac{dy}{dx} = (2x + 3)\frac{d}{dx}(x^3 - 7x + 4)$$

$$+ (x^3 - 7x + 4)\ \frac{d}{dx}\ (2x + 3)$$

$$= (2x + 3)(3x^2 - 7) + (x^3 - 7x + 4)\ (2)$$

$$= 6x^3 + 9x^2 - 14x - 21 + 2x^3 - 14x + 8$$

$$= 8x^3 + 9x^2 - 28x - 13$$

iii) $y = x.\log x$

Differentiating *w.r.t.x*

$$\frac{dy}{dx} = x\frac{d}{dx}\ (\log x) + \log x\ \frac{d}{dx}(x)$$

$$= x\frac{1}{x} + \log x.1 \quad = 1 + \log x$$

iv) $y = \log x\ (e^x + x)$

Differentiating *w.r.t.x*

$$\frac{dy}{dx} = \log x\frac{d}{dx}(e^x + x) + (e^x + x)\ \frac{d}{dx}(\log x)$$

$$= \log x\left(\frac{d}{dx}(e^x) + \frac{d}{dx}(x)\right) + (e^x + x)\ \left(\frac{1}{x}\right)$$

$$= \log x(e^x + 1) + (e^x + x)\frac{1}{x}$$

$$= e^x\left(\log x + \frac{1}{x}\right) + \log x + 1$$

2) Differentiate the following functions *w.r.t.x*

i) $y = \dfrac{x^2 + 7x - 9}{x^2 - 1}$

ii) $y = \dfrac{x}{x + 3}$

iii) $y = \dfrac{1 + \log x}{x}$

Solution:

i) $y = \dfrac{x^2 + 7x - 9}{x^2 - 1} = \dfrac{N}{D}$, say.

Differentiating *w.r.t.x*.

$$\frac{dy}{dx} = \frac{D.\frac{d}{dx}(N) - N\frac{d}{dx}(D)}{D^2}$$

(by quotient rule)

$$= \frac{(x^2 - 1)(2x + 7) - (x^2 + 7x - 9)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^3 - 2x + 7x^2 - 7 - 2x^3 - 14x^2 + 18x}{(x^2 - 1)^2}$$

$$= \frac{-7x^2 + 16x - 7}{(x^2 - 1)^2}$$

ii) $y = \dfrac{x}{x + 3}$

Differentiating *w.r.t.x*.

$$\frac{dy}{dx} = \frac{(x + 3)\frac{d}{dx}x - x\frac{d}{dx}(x + 3)}{(x + 3)^2}$$

$$= \frac{(x+3)\ 1 - x(1)}{(x+3)^2}$$

$$= \frac{3}{(x+3)^2}$$

iii) $y = \dfrac{1+\log x}{x}$

Differentiating *w.r.t.x.*, we get

$$\frac{dy}{dx} = \frac{x\dfrac{d}{dx}(1+\log x) - (1+\log x)\dfrac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(0+\dfrac{1}{x}\right) - (1+\log x)(1)}{x^2}$$

$$= \frac{1-1-\log x}{x^2}$$

$$= \frac{-\log x}{x^2}$$

Derivatives of Logarithmic and Exponential functions

Sr.No.	y	dy/dx
	$f(x)$	$f'(x)$
01	$\log x$	$1/x$
02	e^x	e^x
03	$a^x (a>0)$	$a^x \log a$

EXERCISE 9.1

(I) Find the derivatives of the following functions *w. r. t. x.*

(1) x^{12} (2) x^{-9} (3) $x^{3/2}$

(4) $7x\sqrt{x}$ (5) 3^5

(II) Differentiate the following *w. r. t. x.*

(1) $x^5 + 3x^4$

(2) $x\sqrt{x} + \log x - e^x$

(3) $x^{5/2} + 5x^{7/5}$ (4) $\dfrac{2}{7}x^{7/2} + \dfrac{5}{2}x^{2/5}$

(5) $\sqrt{x}\,(x^2 + 1)^2$

(III) Differentiate the following *w. r. t. x*

(1) $x^3 \log x$ (2) $x^{\frac{5}{2}} e^x$

(3) $e^x \log x$ (4) $x^3.3^x$

(IV) Find the derivatives of the following *w. r. t. x.*

(1) $\dfrac{x^2+a^2}{x^2-a^2}$ (2) $\dfrac{3x^2-5}{2x^3-4}$

(3) $\dfrac{\log x}{x^3-5}$ (4) $\dfrac{3e^x-2}{3e^x+2}$

(5) $\dfrac{xe^x}{x+e^x}$

(V) Find the derivatives of the following functions by the first principle.

i) $3x^2 + 4$ ii) $x\sqrt{x}$

iii) $\dfrac{1}{2x+3}$ iv) $\dfrac{x-1}{2x+7}$

SOMEAPPLICATIONSOFDERIVATIVES:

Supply and **demand** are perhaps one of the most fundamental concepts of economics and it is the backbone of a market economy. **Supply** represents how much the market can offer. The quantity supplied refers to the amount of a certain goods producers are willing to **supply** when receiving a certain price. **Demand** is the quantity of goods or commodity demanded. Similarly **Cost, Revenue and Profit** etc are most commonly used terminologies in business and economics.

DEMAND FUNCTION :

As Demand (D) is a function of Price (P), we can express it as $D = f(P)$.

Marginal Demand (MD):

The rate of change of demand with respect to price is called the **Marginal Demand** (MD).

So, $MD = \dfrac{dD}{dp} = f'(p)$

SUPPLY FUNCTION (S):

Supply (S) is also a function of Price (P), we can express it as $S = g(p)$.

Marginal Supply (MS):

The rate of change of supply with respect to price is called the **Marginal Supply** (MS).

Thus, $MS = \dfrac{dS}{dp} = g'(p)$

TOTAL COST FUNCTION (C):

A Cost (C) function is a mathematical formula used to express the production expenses of number of goods produced. It can be express as $C = f(x)$ where x is the number of goods produced.

Marginal Cost (MC):

The rate of change of Cost with respect to number of goods i.e. x is called **Marginal Cost** (MC).

Therefore, $MC = \dfrac{dC}{dx} = f'(x)$

Average Cost (AC) is the cost of production of each goods. So, $AC = \dfrac{C}{x}$

Revenue and Profit Functions:

If R(x) is the revenue received from the sale of x units of some goods (or commodity), then the derivative, R'(x) is called the Marginal Revenue.

Total Revenue (R):

The Total Revenue is given by R = P.D where P is price and D is quantity of goods demanded.

Average Revenue = $\dfrac{R}{D} = \dfrac{P.D}{D} = P$ i.e. price it self.

Total Profit *(P)* = Revenue – Cost

$$P = R - C$$

SOLVED EXAMPLES

Ex. (1) The demand D of a goods at price p is given by $D = P^2 + \dfrac{32}{P}$.

Find the marginal demand when price is Rs.4.

Solution : Demand is given by $D = P^2 + \dfrac{32}{P}$

Diff. w. r. t. p.

$$\frac{dD}{dP} = 2P - \frac{32}{P^2}$$

$$\left(\frac{dD}{dP}\right)_{p=4} = 2(4) - \frac{32}{16} = 8 - 2 = 6$$

Ex. (2) The cost C of an output is given as $C = 2x^3 + 20x^2 - 30x + 45$. What is the rate of change of cost when the the output is 2 ?

Solution :

$C = 2x^3 + 20x^2 - 30x + 45$

Diff. w. r. t. x.

$$\frac{dC}{dx} = 6x^2 + 40x - 30$$

$$\left(\frac{dC}{dx}\right)_{x=2} = 6(2)^2 + 40(2) - 30 = 74$$

Average Cost = $AC = \dfrac{C}{x}$

$$= \frac{2x^3 + 20x^2 - 30x + 45}{x}$$

$$= 2x^2 + 20x - 30 + \frac{45}{x}$$

Ex. (3) The demand D for a chocolate is inversely proportional to square of it's price P. It is observed that the demand is 4 when price is 4 per chocolate. Find the marginal demand when the price is 4.

Solution : Given that Demand is inversely proportional to square of the price

i.e $D \propto \frac{1}{P^2}$

$D = \frac{k}{P^2}$ where $k > 0$ ...... (1)

Also , D = 4, when P = 4 , from (1)

$$4 = \frac{k}{4^2} \Longrightarrow k = 64$$

Equation (1) becomes

$$D = \frac{64}{P^2}$$

Diff. w. r. t. P.

$\frac{dD}{dP} = -\frac{128}{P^3}$ $\therefore$ when P = 4,

$$\left(\frac{dD}{dP}\right)_{P=4} = -\frac{128}{64} = -2$$

Ex (4) The relation between supply S and price P of a commodity is given as $S = 2P^2 + P - 1$.

Find the marginal supply at price 4.

Solution :

Given , $S = 2P^2 + P - 1$

Diff. w. r. t. P.

$$\therefore \frac{dS}{dP} = 4P + 1$$

$\therefore$ Marginal Supply, when P = 4.

$$\left(\frac{dS}{dP}\right)_{p=4} = 4(4) + 1 = 17$$

EXERCISE 9.2

(I) Differentiate the following functions *w.r.t.x*

(1) $\frac{x}{x+1}$ (2) $\frac{x^2+1}{x}$ (3) $\frac{1}{e^x+1}$

(4) $\frac{e^x}{e^x+1}$ (5) $\frac{x}{\log x}$ (6) $\frac{2^x}{\log x}$

(7) $\frac{(2e^x-1)}{(2e^x+1)}$ (8) $\frac{(x+1)(x-1)}{(e^x+1)}$

II) Solve the following examples:

(1) The demand D for a price P is given as $D = \frac{27}{P}$, find the rate of change of demand when price is 3.

(2) If for a commodity; the price-demand relation is given as $D = \frac{P+5}{P-1}$. Find the marginal demand when price is 2.

(3) The demand function of a commodity is given as $P = 20 + D - D^2$. Find the rate at which price is changing when demand is 3.

(4) If the total cost function is given by; $C = 5x^3 + 2x^2 + 7$; find the average cost and the marginal cost when $x = 4$.

(5) The total cost function of producing n notebooks is given by $C = 1500 - 75n + 2n^2 + \frac{n^3}{5}$. Find the marginal cost at $n = 10$.

(6) The total cost of 't' toy cars is given by $C=5(2^t)+17$. Find the marginal cost and average cost at t=3.

(7) If for a commodity; the demand function is given by, $D = 75-3P$. find the marginal demand function when $P = 5$

(8) The total cost of producing x units is given by $C=20e^x$, find its marginal cost and average cost when $x = 4$

(9) The demand function is given as $P = 175 + 9D + 25D^2$. Find the revenue, average revenue and marginal revenue when demand is 10.

(10) The supply S for a commodity at price P is given by $S = P^2 + 9P - 2$. Find the marginal supply when price is 7.

(11) The cost of producing x articles is given by $C = x^2 + 15x + 81$. Find the average cost and marginal cost functions. Find marginal cost when $x = 10$. Find x for which the marginal cost equals the average cost.

Let's remember!

- If u and v are differentiable functions of x and if

  (i) $y = u \pm v$ then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

  (ii) $y = u.v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

  (iii) $y = \frac{u}{v}$, $v \neq 0$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

- Standard Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1} for all $n \in N$
x	1
$1/x$	$-1/x^2$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
k (constant)	0
e^x	e^x
a^x	$a^x \log a$
$\log x$	$1/x$

MISCELLANEOUS EXERCISE - 9

I. Differentiate the following functions.*w.r.t.x.*

(1) x^5 (2) x^{-2} (3) $\sqrt{x}$

(4) $x\sqrt{x}$ (5) $\frac{1}{\sqrt{x}}$ (6) 7^x

II. Find $\frac{dy}{dx}$ if

(1) $y = x^2 + \frac{1}{x^2}$ (2) $y = \left(\sqrt{x}+1\right)^2$

(3) $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

(4) $y = x^3 - 2x^2 + \sqrt{x} + 1$

(5) $y = x^2 + 2^x - 1$

(6) $y = (1-x)(2-x)$

(7) $y = \frac{1+x}{2+x}$ (8) $y = \frac{(\log x + 1)}{x}$

(9) $y = \frac{e^x}{\log x}$

(10) $y = x \log x (x^2 + 1)$

III. Solve the following.

(1) The relation between price (P) and demand (D) of a cup of Tea is given by $D = \frac{12}{P}$. Find the rate at which the demand changes when the price is Rs. 2/- Interpret the result.

(2) The demand (D) of biscuits at price P is given by $D = \frac{64}{P^3}$, find the marginal demand when price is Rs. 4/-.

(3) The supply S of electric bulbs at price P is given by $S = 2P^3 + 5$. Find the marginal supply when the price is Rs. 5/- Interpret the result.

(4) The marginal cost of producing x items is given by $C = x^2 + 4x + 4$. Find the average cost and the marginal cost. What is the marginal cost when $x = 7$.

(5) The Demand D for a price P is given as $D = \frac{27}{P}$, Find the rate of change of demand when the price is Rs. 3/-.

(6) If for a commodity; the price demand relation is given be $D = \left(\frac{P+5}{P-1}\right)$. Find the marginal demand when price is Rs. 2/-.

(7) The price function P of a commodity is given as $P = 20 + D - D^2$ where D is demand. Find the rate at which price (P) is changing when demand $D = 3$.

(8) If the total cost function is given by $C = 5x^3 + 2x^2 + 1$; Find the average cost and the marginal cost when $x = 4$.

(9) The supply S for a commodity at price P is given by $S = P^2 + 9P - 2$. Find the marginal supply when price Rs. 7/-.

(10) The cost of producing x articles is given by $C = x^2 + 15x + 81$. Find the average cost and marginal cost functions. Find the marginal cost when $x = 10$. Find x for which the marginal cost equals the average cost.

Gottfried Wilhelm Leibniz (1646 – 1716)

Gottfried Wilhelm Leibniz was a prominent German polymath and philosopher in the history of mathematics and the history of philosophy. His most notable accomplishment was conceiving the ideas of differential and integral calculus, independently of Isaac Newton's contemporaneous developments. Calculus was discovered and developed independently by Sir Isaac Newton (1642 - 1727) in England and Gottfried Wilhelm Leibniz (1646 - 1716) in Germany, towards the end of 17th century.

ACTIVITIES

Activity 9.1:

Find the derivative of $\frac{1}{x^2}$ by first principle.

$f(x) = \frac{1}{x^2}$, $\quad f(x+h) = \frac{1}{(x+h)^2}$

$$f'(x) = \lim_{h \to 0}\left(\frac{\square - \square}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right)$$

$$= \lim_{h \to 0}\frac{\square - \square}{h(x+h)^2 x^2}$$

$$= \lim_{h \to 0}\frac{x^2 - (x^2 + 2xh + h^2)}{h(x+h)^2 . \square}$$

$$= \lim_{h \to 0} \frac{\boxed{}}{h(x+h)^2 . x^2}$$

$$= \frac{-\boxed{}}{\boxed{}} = \_\_\_\_\_\_$$

Activity 9.2:

The supply S of pens at price P is given by $S = 2P^3 + 7$. Find the marginal supply when price is Rs.5/-. Interpret the result.

Given : $S = 2p^3 + 7$

$$\therefore \quad \frac{ds}{dp} = \text{............} = \text{............}$$

$\therefore$ Marginal supply when P = 5/-

$$= \left(\frac{ds}{dp}\right)_{p=5} = \boxed{} = \boxed{}$$

Interpretation : ____________________

Activity 9.3:

The relation between price (P) and demand (D) at a cup of Tea is given as $D = \frac{32}{p}$. Find the rate of which the demand changes when the price is Rs. 4/-. Interpret the result.

Given : $D = \frac{32}{p}$

$$\therefore \quad \frac{dD}{dP} = \text{............} = \text{............}$$

$$= \left(\frac{dD}{dp}\right)_{P=4} = \boxed{}$$

Interpretation : ____________________

❖ ❖ ❖

ANSWERS

1. SETS AND RELATIONS

Exercise 1.1

1) i) A = {M, A, R, I, G, E}

   ii) B = {0, 1, 2, 3, 4}

   iii) C = {2, 4, 6, 8, ...}

2) i) $\{x/x \in W, x \notin N\}$

   ii) $\{x/-3 \leq x \leq 3, x \in Z\}$

   iii) $\{x/x = \frac{n}{n^2+1}, n \in N, n \leq 7\}$

3) i) $A \cup B \cup C = \{\frac{-5}{3}, -1, -\frac{1}{2}, \frac{3}{2}, 3\}$

   ii) $A \cap B \cap C = \{\ \}$

6) i) 45 ii) 10 iii) 10 iv) 25

7) i) 132 ii) 63

8) i) 1750 ii) 250 iii) 1100

9) 42

10) i) 114 ii) 38 iii) 188

11) P(A) = {{1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3}, {ϕ}}

12) i) $\{x / x \in R \ , \ -3 < x < 0\}$

   ii) $\{x / x \in R \ , \ 6 \leq x \leq 12\}$

   iii) $\{x / x \in R \ , \ 6 < x \leq 12\}$

   iv) $\{x / x \in R \ , \ -23 \leq x < 5\}$

Exercise 1.2

1) $x = 2, y = -2$

2) $x = 0, y = \frac{15}{2}$

3) i) A×B = {(*a*,*x*), (*a*,*y*), (*b*,*x*), (*b*,*y*), (*c*,*x*), (*c*,*y*)}

   ii) B×A = {(*x*,*a*) , (*x*,*b*), (*x*,*c*), (*y*,*a*), (*y*,*b*), (*y*,*c*)}

   iii) A×A = {(*a*,*a*) , (*a*,*b*), (*a*,*c*), (*b*,*a*), (*b*,*b*), (*b*,*c*), (*c*,*a*), (*c*,*b*), (*c*,*c*)}

   iv) B×B = {(*x*,*x*) , (*x*,y), (y,x), (y,y)}

4) i) P×Q = {(1,6), (2,6), (3,6) (1,4), (2,4), (3,4)}

   ii) Q×P = {(6,1), (6,2), (6,3) (4,1), (4,2), (4,3)}

5) i) A×(B∩C) = {(1,5), (1,6), (2,5), (2,6), (3,5), (3,6), (4,5), (4,6)}

   ii) {(1,5), (1,6), (2,5), (2,6), (3,5), (3,6), (4,5), (4,6)}

   iii) {(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6)}

   iv) {(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6)}

6) {(0,10), (6,8), (8,6), (10,0)}

7) i) Domain = {1,2,3,4,5}; Range = {4}

   ii) Domain = {1,2,3,4,5,6,7,8,9,10,11};
   Range = {11,10,9,8,7,6,5,4,3,2,1)}

   iii) Domain = {2}; Range = {4,5,6,7}

9) i) R_1 = {(2,4), (3,9), (5,25), (7,49),(11,121), (13,169)}

Domain = {2,3,5,7,11,13}

Range = {4,9,25,49,121,169}

ii) $R_2 = \{(1,1), (2,\frac{1}{2}), (3,\frac{1}{3}), (4,\frac{1}{4}), (5,\frac{1}{5})\}$

Domain = {1,2,3,4,5}

Range = $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$

10) Range = {2,3,4,5}

11) i) {(1,3), (2,6), (3,9)}

ii) {(1,4), (1,6), (2,4), (2,6)}

iii) {(0,3), (1,2), (2,1), (3,0)}

MISCELLANEOUS EXERCISE - 1

1) i) A = {x/x = 10n, $n \in$ N, $n \le 5$}

ii) B = {x/x is vowel of English alphabets}

iii) C = {x/x represents day of a week}

2) i) A∪B = {1,2,4,6,7,10,11}

ii) B∩C = {} = φ

iii) A-B = {1,10}

iv) B–C = {2,4,6,7,11}

v) A∪B∪C = {1,2,3,4,5,6,7,8,9,10,11,12}

vi) A∩ (B∪C) = {4,7}

3) 230

4) 12

5) i) A×B = {(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)}

B×A = {(2,1), (2,2), (2,3), (4,1), (4,2), (4,3)}

A×A = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)}

B×B = {(2,2), (2,4), (4,2), (4,4)

(A×B) ∩ (B×A) = {(2,2)}

ii) A×A×A = {(–1,–1,–1), (–1,–1,1), (–1,1,–1), (1,–1,–1), (1,–1,1), (1,1,–1), (1,1,1), (–1,1,1)}

6) i) R_1 is a relation

ii) R_2 is a relation

iii) R_3 is a relation

iv) R_4 is not a relation

7) Domain = {1,2,3,4}

Range = {4}

2. FUNCTION

Exercise 2.1

1) a) It is a function

b) It is not a function

c) It is not a function

2) a) It is not a function

b) It is a function

c) It is not a function

d) It is a function

3) a) 1 b) 19 c) $-\frac{1}{4}$ d) $x^2 - x - 1$

e) $x^2 + 3x + 1$

4) a) $\frac{6}{5}$ b) $x = \pm 3$

c) $x = \frac{1}{2}$ or $x = \frac{-2}{3}$

5) $x = 0$ or $x = \pm 3$

6) a) $f(3) = 22$ b) $f(2) = 7$ c) $f(0) = 3$

7) a) $f(-4) = -18$ b) $f(-3) = -14$
c) $f(1) = 5$ d) $f(5) = 25$

8) a) $9x + 4$ b) 0
c) 238 d) $\frac{3x+5}{6x-1}$ domain = $R - \{\frac{1}{6}\}$

9) a) $50x^2 - 40x + 11$ b) $10x^2 + 13$
c) $8x^4 + 24x^2 + 21$ d) $25x - 12$

MISCELLANEOUS EXERCISE - 2

1) i) Yes, Domain = {2,4,6,8,10,12,14}
Range = {1,2,3,4,5,6,7}
ii) Not a function
iii) Yes, Domain = {1,3,5},
Range = {1,2}

2) $f^{-1}(x) = \frac{5(x-2)}{3}$

3) $f(-1) = 1$
$f(-2) = -3$
$f(0)$ = does not exist

4) 2

5) $3x^2 - 11x + 15$

6) $a = 4$, $f(4) = 16$

7) $a = 3$, $b = -2$

3. COMPLEX NUMBERS

Exercise 3.1

1) i) $3 - i$ ii) $3 + i$
iii) $-\sqrt{5} + \sqrt{7}\,i$ iv) $\sqrt{5}\,i$
v) $-5i$ vi) $\sqrt{5} + i$
vii) $\sqrt{2} - \sqrt{3}\,i$

2) i) $a = -4$, $b = -3$ ii) $a = \frac{-7}{2}$, $b = \frac{1}{2}$
iii) $a = \frac{3}{10}$, $b = \frac{-1}{10}$ iv) $a = \frac{-8}{29}$, $b = 0$
v) $a = \frac{11}{19}$, $b = \frac{2\sqrt{3}}{19}$
vi) $a = 13$, $b = 0$
vii) $a = \frac{23}{13}$, $b = \frac{15}{13}$

4) i) $-i$ ii) 1 iii) i
iv) 1 v) $-i$ vi) -1
vii) 0

6) i) $2i$ ii) 0

7) 1

8) i) $x = 1$, $y = 2$
ii) $x = -2$, $y = 2$

9) i) 7
ii) 2

Exercise 3.2

1) i) $\pm(1 - 3i)$ ii) $\pm(4 + 3i)$
iii) $\pm(2 + \sqrt{3}\,i)$ iv) $\pm(\sqrt{5} + \sqrt{2}\,i)$
v) $\pm(\sqrt{3} - i)$

2) i) $\frac{-1 \pm \sqrt{7}\,i}{8}$ ii) $\frac{\sqrt{3} \pm \sqrt{5}\,i}{4}$
iii) $\frac{7 \pm \sqrt{11}\,i}{6}$ iv) $2 \pm 3i$

3) i) $x = 2i$ or $x = -5i$
ii) $x = \frac{1}{2}i$ or $x = -2i$
iii) $x = -2i$ iv) $x = -2i$

4) i) $x = 3 - i$ or $x = -1 + 2i$

ii) $x = 3\sqrt{2}$ or $x = 2i$

iii) $x = 3 - 4i$ or $x = 2 + 3i$

iv) $x = 1 - i$ or $x = \frac{4}{5} - \frac{2}{5}i$

Exercise 3.3

1) i) 7 ii) 65 iii) w^2

2) i) – 1 ii) 0 iii) – 1

iv) 0 v) 1

MISCELLANEOUS EXERCISE - 3

1) –1

2) $-3\sqrt{2}$

3) i) $3 + 8i$ ii) $-4 + 0i$

iii) $14 - 5i$ iv) $\frac{15}{2} - 10i$

v) $-30 + 10i$ vi) $\frac{1}{2} + \frac{7}{2}i$

vii) $\frac{-35}{26} - \frac{45}{26}i$ viii) $\frac{1}{4} + i\frac{\sqrt{15}}{4}$

ix) $-i$ x) $\frac{8}{5} + \frac{56}{25}i$

4) i) $x = 2, y = 1$ ii) 3,2

iii) 17,19 iv) $\frac{28}{61}, \frac{3}{61}$

v) 4,–2

5) i) 1 ii) –2 iii) –3

6) i) $\pm(3 + 5i)$ ii) $\pm(4 - i)$

iii) $\pm(\sqrt{3} + i)$ iv) $\pm(3 + 3i)$

v) $\pm(2 - i)$ vi) $\pm\sqrt{2}(2 + i)$

4. SEQUENCES AND SERIES

Exercise 4.1

1) i) $t_n = 2(3^{n-1})$ ii) $t_n = (-5)^{n-1}$

iii) $t_n = (5)^{3/2 - n}$ iv) it is not a G.P.

v) it is not a G.P.

2) i) $t_7 = \frac{1}{81}$ ii) $t_3 = \frac{7}{2187}$

iii) $t_6 = -1701$ iv) $r = 3$

3) $t_{10} = 5^{10}$

4) $x = \pm\frac{4}{9}$

5) G.P. with $a = \frac{4}{25}$, $r = \frac{5}{2}$

6) 3,6,12 and 12,6,3

7) $\frac{1}{27}, \frac{1}{3}, 3, 27$ or $27, 3, \frac{1}{3}, \frac{1}{27}$

8) 1, 2, 4, 8, 16 or 1, -2, 4, -8, 16

Exercise 4.2

1) i) $S_n = 3(2^n - 1)$

ii) $S_n = \frac{p^{2-n}\left(q^n - p^n\right)}{q - p}$

2) i) $S_6 = \frac{266}{243}$

ii) $a = 3$

3) i) $n = 5$

ii) $r = \frac{3}{5}$

4) i) 635

ii) $S_{10} = 2046$

5 i) $\frac{1}{3}\{\frac{10}{9}(10^n - 1) - n\}$

ii) $\frac{8}{9}\{\frac{10}{9}(10^n-1)-n\}$

6 i) $\frac{4}{9}\{n-\frac{1}{9}$ $[1-(0.1)^n\}$

ii) $\frac{7}{9}\{n-\frac{1}{9}$ $[1-(0.1)^n\}$

7 i) $t_n=\frac{5}{9}[1-(0.1)^n]$

ii) $t_n=\frac{2}{9}\{1-(0.1)^n\}$

8) $t_n=\frac{4}{3}(3^n)$

Exercise 4.3

1) i) Sum to infinity = 1

ii) Sum to infinity = 6

iii) $\frac{-9}{4}$

iv) Sum to infinity does not exists.

2) i) $0.\overline{32}=\frac{32}{99}$

ii) $3.\overline{5}=\frac{32}{9}$

iii) $4.\overline{18}=\frac{46}{11}$

iv) $0.3\overline{45}=\frac{342}{990}=\frac{19}{55}$

v) $3.4\overline{56}=\frac{3422}{990}=\frac{1711}{495}$

3) $a=4$

4) $r=\frac{6}{11}$

5) $\frac{15}{4},\frac{15}{16},\frac{15}{64},\ldots\ldots\ldots\ldots\ldots\ldots$

Exercise 4.4

1) i) Given series is a H.P..

ii) Given series is a H.P.

iii) Given series is a H.P.

2) i) $\frac{1}{3n-1},\frac{1}{23}$

ii) $\frac{1}{2n+2},\frac{1}{18}$

iii) $\frac{1}{5n},\frac{1}{40}$.

3) A=5

4) $H=\frac{24}{5}$

5) G =60

6) $\frac{1}{9}$ and $\frac{1}{11}$

7) −3 and 9

8) 4 and 9

9) 14 and 56

Exercise 4.5

1) $\frac{n(4n^2+9n-1)}{6}$

2) $\frac{n(2n^2+n+1)}{2}$

3) $\frac{n(n+3)}{4}$

4) $\frac{n(n+1)(n+2)}{12}$

5) $\frac{n(16n^2+48n+41)}{3}$

6) $\frac{2n(n+1)(2n+1)}{3}$

7) 2485

8) $n(6n^3 + 8n^2 + 3n - 2)$

9) $n = 48$

MISCELLANEOUS EXERCISE - 4

1) $t_{10} = 3072.$

2) $r = \frac{3}{4}$

3) $a = \frac{49}{5}, r = \frac{5}{7}$

4) 5,10,20 or 20, 10, 5

5) $\frac{1}{27}, \frac{1}{3}$, 3,27 or 27, 3, $\frac{1}{3}, \frac{1}{27}$

6) $\frac{1}{3}$, 1, 3, 9, 27, or 27, 9, 3, 1, $\frac{1}{3}$

7) The sequence is a G.P. r = 7

8) $\frac{2}{9}\left[\frac{10}{9}(10^n-1) - n\right]$

9) $t_n = \frac{2}{3}\left[1-(0.1)^n\right]$

10) $\frac{n(10n^2+27n-1)}{6}$

11) $\frac{n(n+1)(3n^2-17n+26)}{12}$

12) $\frac{n(n+1)(n+2)}{18}$

13) $\frac{n(n+1)(2n+1)}{24}$

14) $2n(n+1)(n+2)$

15) 2364

16) 1275

17) $r = \pm 15$

18) $k=2$

19) 1

5. STRAIGHT LINE

Exercise 5.1

1. $2x - 4y + 5 = 0$
2. $9x - y + 6 = 0$
3. $3x^2 + 3y^2 + 4x - 24y + 32 = 0$
4. $x^2 + y^2 - 11x - 11y + 53 = 0$
5. $3x + 4y - 41 = 0$
6. $x^2 + y^2 - 4x - 11y + 33 = 0$
7. (a) (–1,0) (b) (0,2)
8. (a) (6,7) (b) (4,6)
9. (–3, 11)
10. (a) $3X - Y + 6 = 0$
    (b) $X^2 + Y^2 + X + 4Y - 5 = 0$
    (c) $XY = 0$

Exercise 5.2

1. a) Slope of the line AB = 2
   b) Slope of the line CD = $\frac{4}{7}$
   c) Its slope is not defined.
   d) Slope of the line is 0.
2. $-\frac{3}{2}$
3. $\frac{1}{\sqrt{3}}$
4. 1
5. –1.
7. 1
8. $k = 1$

Exercise 5.3

1. a) $y = 5$ b) $x = -5$ c) $y = -1$ and $y = 7$
2. a) $y = 3$ b) $x = 4$
3. a) $x = 2$ b) $y = -3$
4. $4x - y - 8 = 0$
5. m = 1, c = –1
6. a) $2x + y - 4 = 0$
   b) $2x - 5y + 14 = 0$
   c) $2x + 4y - 13 = 0$.
7. a) X- intercept 3, Y-intercept 2
   b) X- intercept $\frac{2}{3}$, Y-intercept $\frac{3}{2}$
   c) X- intercept –6, Y-intercept 4
8. $x + y - 7 = 0$
9. a) $5x + y - 15 = 0$
   b) $3x + 4y - 14 = 0$
   c) $2x - 3y - 1 = 0$

Exercise 5.4

1. a) slope $-\frac{2}{3}$, X-intercept 3, *Y*-intercept 2
   b) slope $-\frac{1}{2}$, both the intercepts 0
2. a) $2x - y - 4 = 0$, b) $0x + 1y - 4 = 0$
   c) $2x + y - 4 = 0$ d) $2x - 3y + 0 = 0$
4. $P = \pm 24$
5. (1, –1)
6. $x + 3y = 3$
7. 4 units
8. $\frac{25}{\sqrt{117}}$ units
9. $8x + 13y - 24 = 0$
10. $2x + y + 13 = 0$, $x - 9y + 73 = 0$, $11x - 4y - 52 = 0$, $\left(\frac{-1}{19}, \frac{-10}{19}\right)$

MISCELLANEOUS EXERCISE - 5

1. a) $-\frac{7}{2}$ b) $-\frac{1}{4}$ c) –1 d) 4
2. a) $\frac{1}{\sqrt{3}}$ b) $\frac{4}{3}$ c) $-\frac{1}{2}$
3. a) 22 b) $\frac{5}{3}$ c) 1
4. $y = -2x - \frac{8}{3}$, slope = –2.
6. 1
7. 1
8. No, point does not satisfy the equation.
9. (d) $2x - y = 0$.
10. a) $y + 3 = 0$ b) $x + 2 = 0$
    c) $y = 5$ d) $x = 3$
11. a) $y = 3$ b) $y = 4$
    c) $x = 2$
12. a) $5x - y + 7 = 0$ b) $13x - y = 25$
    c) $x = 7$ d) $x = 0$
    e) $3x - 2y = 0$
13. $4x - 3y + 12 = 0$
14. a) $5x - y - 25 = 0$ b) $\sqrt{3}x - y + 4 = 0$
15. a) BC : $3x + y = 9$, CA : $x = 1$,
    AB : $x + y = 5$
    b) Median AD : $x - y + 3 = 0$,
    Median BE : $2x + y = 7$,
    Median CF: $5x + y = 11$
    c) $x - 3y + 12 = 0$, $y = 5$, $x - y + 2 = 0$,
    d) $x - 3y + 11 = 0$, $y = 3$, $x - y + 5 = 0$

6. DETERMINANTS

Exercise 6.1

1) i) 49 ii) -358
iii) $-27+9i$ iv) -20
v) -10 vi) 46
vii) $abc + 2fgh - af^2 - bg^2 - ch^2$
viii) 0

2) i) $x = 2$ ii) $x = \frac{14}{5}$
iii) $x = 1$ or $x = 2$ or $x = 3$

3) i) $x = 2$ or $x = -4$ ii) $x = -1$ or $x = 2$

4) $x = -2$

5) $x = 11$ and $y = 52$

Exercise 6.2

1) i) 0 ii) 0 iii) 0

2) $4abc$

3) $x = -\frac{7}{3}$

4) $x = 0$, or $x = 12$

5) $10\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 7 \\ 3 & 2 & 6 \end{vmatrix}$

6) i) 0 ii) 0

7) (i) $\begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Exercise 6.3

1) i) $x = \frac{5}{3}$, $y = 1$, $z = \frac{-4}{3}$

ii) $x = \frac{1499}{447}$, $y = \frac{520}{447}$, $z = \frac{332}{447}$

iii) $x = 4$, $y = 7$, $z = 6$

iv) $x = \frac{3}{5}$, $y = \frac{-3}{5}$, $z = \frac{-1}{2}$

v) $x=1$, $y = -2$, $z= 2$

2) Rs. 1750 , Rs. 1500, Rs. 1750

3) Consistent

4) i) $k = 16$ ii) $k = \frac{22}{5}$

5) i) 13 sq.unit ii) $\frac{35}{2}$ sq. unit
iii) 25 sq. unit

6) $k = 3$; $k = \frac{-7}{3}$

7) $\frac{35}{2}$ sq. unit

8) $A(\Delta PQR) = 0$

9) 3,5,7 are the three required numbers

MISCELLANEOUS EXERCISE - 6

1) i) -113 ii) -76

2) i) $x = \frac{-1}{3}$ or $x = 2$ ii) $x = \frac{2}{3}$

3) 0

4) i) 0 ii) LHS = RHS
iii) LHS = RHS iv) 0

5) i) $x = 1$, $y = 2$, $z = 1$

ii) $x = \frac{3}{5}$, $y = \frac{-3}{5}$, $z = \frac{-1}{2}$

iii) $x = \frac{9}{2}$, $y = -\frac{3}{2}$, $z = \frac{1}{2}$

6) i) $k = 5$ ii) $k = \frac{14}{5}$ or $k = 2$

7) i) 4 sq. unit

ii) $\frac{25}{2}$ sq. unit iii) $\frac{13}{2}$ sq. unit

8) i) $k = 0$; $k = 8$ ii) $k = 34$; $k = 1$

7. LIMITS

Exercise 7.1

I. 1. 1 2. $\frac{-3}{16}$ 3. $\frac{3}{125}$ 4. $\pm\frac{2}{\sqrt{3}}$

II. 1. $\frac{2}{3(\sqrt[3]{7})}$ 2. 4 3. 4

III. 1. $\frac{-1}{6}$ 2. 24 3. $\frac{3}{2}$ $(a+2)^{1/2}$

4. $\frac{15}{2}$

Exercise 7.2

I. 1. $\frac{-1}{4}$ 2. $\frac{-1}{2}$ 3. $\frac{-1}{2}$ 4. $\frac{-1}{2}$

II. 1. $\frac{4}{3}$ 2. 0 3. 0

III. 1. 44 2. 3 3. –3 4. 8

Exercise 7.3

I. 1. $\frac{1}{2\sqrt{6}}$ 2. – 1 3. $\sqrt{2}$

II. 1. $\frac{2}{3\sqrt{3}}$ 2. – 8

III. 1. $\frac{7}{2}$ 2. 1 3. 24 4. –24

IV. 1. 2 2. $\frac{-1}{3}$

Exercise 7.4

I. 1. $\frac{1}{log 4}\log\left(\frac{9}{5}\right)$ 2. $\log\left(\frac{15}{2}\right)$ 3. 1

II. 1. $(log 3)^2$ 2. $e^{\frac{2}{3}}$ 3. $\frac{-2}{3}$

III. 1. $\frac{1}{4}\log\left(\frac{a^3}{b^2}\right)$ 2. $\frac{(\log 2)^2}{\log 3}$

3. $(log 3)(log 5)$ 4. $\frac{1}{6}$

IV. 1. $(log 5)^2$ 2. $(log 7 - log 5)^2$

MISCELLANEOUS EXERCISE - 7

I. 1) $n = 5$

II. 1) $\frac{5}{3}(a+2)^{2/3}$ 2) n 3) 1

4) $\frac{3}{7}$ 5) 1 6) $-\frac{1}{3}$ 7) log5

8) $e^{\frac{1}{5}}$ 9) 9 10) $\frac{5}{3}$

11) log(abc) 12) 1 13) 1

14) 2 $(\log a)^2$ 15) $(\log 5)^2$ 16) $\frac{2\log a}{\log b}$

17) 100 18) $\frac{-1}{2}$ 19) 3

8. CONTINUITY

Exercise 8.1

1) i) Continuous at $x = -2$
   ii) Continuous on R except at $x = 3$

2) i) Discontinuous at $x = 2$
   ii) Continuous at $x = 1$

3) i) Discontinuous at $x = 2$
   ii) Continuous at $x = 2$
   iii) Continuous at $x = \frac{8}{3}$
   iv) Continuous at $x = 3$

4) i) $k = \frac{3}{2}$
   ii) $k = (\log 5)^2$
   iii) $a = 2,\ b = -4$
   iv) $a = \frac{1}{2}$, $b = \frac{1}{2}$

MISCELLANEOUS EXERCISE - 8

I) 1) Continuous on its domain except at $x = 5$
   2) Continuous
   3) Continuous
   4) Discontinuous
   5) Discontinuous

II) 1) $k = e^6$, 2) $k = 125$ 3) $k = \frac{3}{2}$

III) 1) $a = 1$, $b = -1$
   2) $a = -1$, $b = -22$
   3) $a = \frac{1}{3}$, $b = \frac{3}{2}$

9. DIFFERENTIATION

Exercise 9.1

I) 1) $12x^{11}$ 2) $-9x^{-10}$
   3) $\frac{3}{2}\sqrt{x}$ 4) $\frac{21}{2}\sqrt{x}$
   5) 0

II) 1) $5x^4 + 12x^3$ 2) $\frac{3\sqrt{x}}{2} + \frac{1}{x} - e^x$
   3) $\frac{5x^{3/2}}{2} + 7x^{2/5}$ 4) $x^{5/2} + x^{-3/5}$
   5) $\frac{9}{2}x^{7/2} + 5x^{3/2} + \frac{1}{2\sqrt{x}}$

III) 1) $x^2 + 3x^2\log x$ 2) $\left(x^{5/2} + \frac{5}{2}x^{3/2}\right)e^x$
   3) $e^x\left(\frac{1}{x} + \log x\right)$ 4) $3^x x^2 (x\log 3 + 3)$

IV) 1) $\frac{-4a^2x}{(x^2 - a^2)^2}$
   2) $\frac{-6x^4 + 30x^2 - 24x}{(2x^3 - 4)^2}$
   3) $\frac{(x^3 - 5)\frac{1}{x} - \log x.3x^2}{(x^3 - 5)^2}$
   4) $\frac{12e^x}{(3e^x + 2)^2}$
   5) $e^x\left[\frac{(x + e^x)(x + 1) - x(1 + e^x)}{(x + e^x)^2}\right]$

V) 1) $6x$ 2) $\frac{3\sqrt{x}}{2}$
   3) $\frac{-2}{(2x + 3)^2}$ 4) $\frac{9}{(2x + 7)^2}$

Exercise 9.2

I) 1) $\frac{1}{(x+1)^2}$ 2) $1-\frac{1}{x^2}$

3) $\frac{-e^x}{(e^x+1)^2}$ 4) $\frac{e^x}{(e^x+1)^2}$

5) $\frac{\log x-1}{(\log x)^2}$ 6) $\frac{2^x(\log x\log 2)-\frac{1}{x}}{(\log x)^2}$

7) $\frac{4e^x}{(2e^x+1)^2}$ 8) $\frac{(2x+1-x^2)e^x+2x}{(e^x+1)^2}$

II) 1) −3 2) −6 3) −5

4) $\frac{dc}{dx}=256;\ AC=\frac{359}{4}$

5) 25

6) MC = 40log2; AC = 19

7) −3

8) $\frac{dc}{dx}=20.e^4;\ AC=5e^4$

9) R = 27650, A.R. = 2765, M.R. = 7855

10) 23

11) $AC = x + 15 + \frac{81}{x}$, $MC = 2x + 15$.

at $x = 10$, $MC = 35$ For $AC = MC$ $x = 9$

MISCELLANEOUS EXERCISE - 9

I. 1) $5x^4$ 2) $\frac{-2}{x^3}$

3) $\frac{1}{2\sqrt{x}}$ 4) $\frac{3}{2}x^{1/2}$

5) $-\frac{1}{2x^{\frac{3}{2}}}$ 6) $7^x\log 7$

II. 1) $2x-\frac{2}{x^3}$ 2) $\left(1+\frac{1}{\sqrt{x}}\right)$

3) $1-\frac{1}{x^2}$ 4) $3x^2-4x+\frac{1}{2\sqrt{x}}$

5) $2x + 2^x\log 2$ 6) $-3 + 2x$

7) $\frac{1}{(2+x)^2}$ 8) $\frac{-\log x}{x^2}$

9) $\frac{e^x(x\log x-1)}{x(\log x)^2}$

10) $2x^2\log x + (x^2 + 1) + (x^2 + 1)\log x$

III. 1) −3. The rate of change of demand is negative it means, the demand will fall when the price becomes Rs. 2/-.

2) $\frac{-3}{4}$, The rate of change of demand is negative means, the demand falls when the price becomes Rs. 4/-.

3) 150, The rate of change of supply w.r.t. price is positive means, supply will increase if the price increase.

4) $AC = x+4+\frac{4}{x}$; $MC = 18$

5) −3 6) −6

7) −5

8) $\frac{dc}{dx} = 256$, $AC = \frac{353}{4}$

9) 23

10) $AC = x + 15 + \frac{81}{x}$,

$MC = 2x + 15$ at $x = 10$

$MC = 35$ for $AC = MC$, $x = 9$

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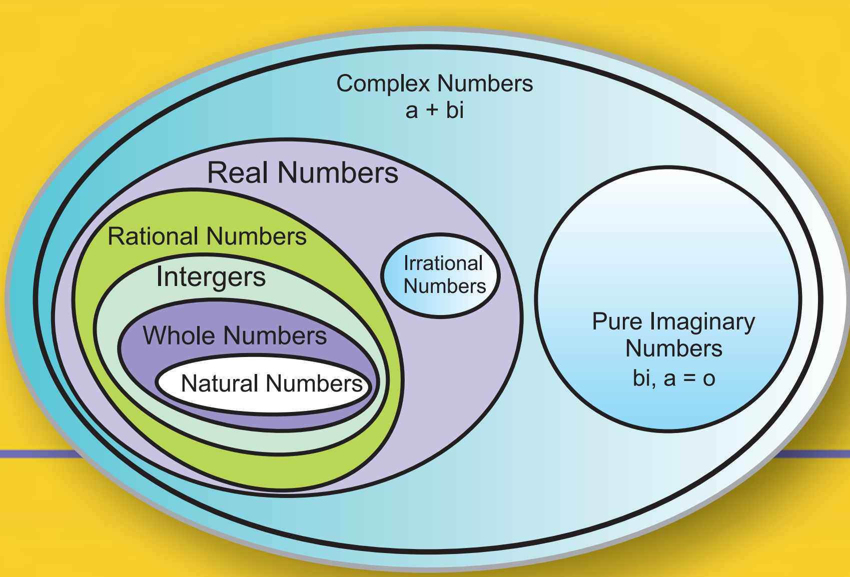

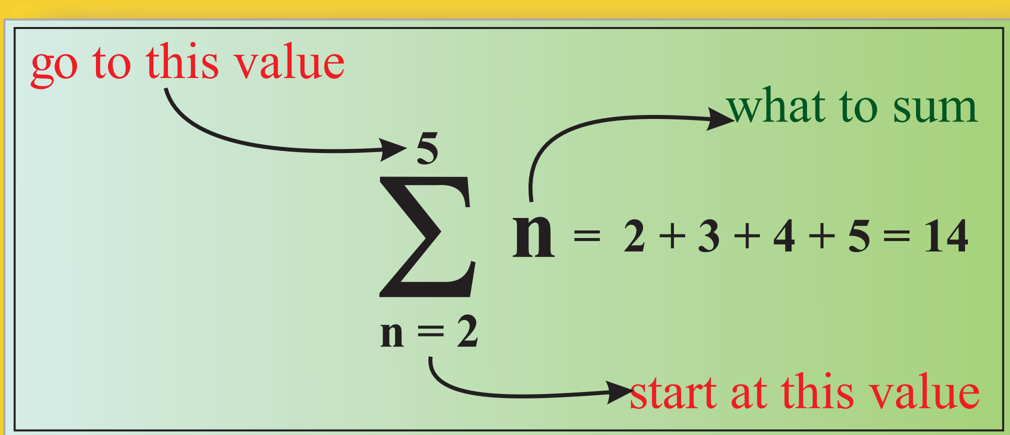

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